Static and Dynamic Correlation Functions in One-Dimensional Bose Liquids

by

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Abstract

This Thesis discusses approaches to calculating static and dynamic correlation functions in one-dimensional gases of interacting bosons. The first part of the Thesis deals with the momentum-momentum correlation function in the weakly interacting 1D Bose gas. In the regime of phase fluctuating quasicondensate this correlator is found to differ qualitatively from the form predicted by Bogoliubov theory of the true condensate in that correlations between any two values of momentum become finite. Linear hydrodynamics used to calculate the quasicondensate correlation function prove to be adequate. A classical field approximation is used to smoothly interpolate between the quasicondensate and strongly degenerate gas regimes.

In the second part the focus is shifted to the dynamical structure factor and it is shown that hydrodynamics is generally inapplicable to calculating the dynamical correlators. The hydrodynamic treatment is enhanced using impurity theory and an exact model-dependent boundary is obtained for the region of applicability of hydrodynamics on the momentum-energy plane. Numerical estimates show that non-hydrodynamic behavior should be observable for the currently available values of experimental parameters.

Contents

Chapter 1

Introduction

The field of ultracold atomic gases was kick-started in 1995 when the combination of advances in laser cooling [1], magnetic trapping [2], evaporative cooling [3] and gas imaging techniques made the experimental realization of the Bose-Einstein condensation (BEC) of alkali atoms possible. The work was done by the groups of Eric Cornell and Carl Wieman at JILA [4], Randy Hulet at Rice University [5], and Wolfgang Ketterle at MIT [6]. This discovery gave access to an entirely new class of experimental systems, indeed a new state of matter, and was celebrated by the 2001 Nobel Prize in Physics awarded to Cornell, Wieman and Ketterle.

Ultracold atomic gases are remarkable experimental systems because the form and depth of traps used to confine them, as well as strength of inter–atomic interactions can be fine-tuned. For example, in an optical lattice the neutral alkali atoms are placed in an intense standing wave of light created by two counter–propagating laser beams and are subject to a periodic potential due to the Stark effect [7]. By varying the intensity and wavelength of the lasers it is possible to tune the depth of the lattice. Another option is magnetically trapping the atoms in certain hyperfine state(s) with the help of correct configuration of magnetic field gradients. What is especially relevant to the present work, magneto-optical lattices allow one

to experimentally produce and study low–dimensional systems: when subjected to a standing wave in one direction, the optical lattice "dices" the atomic cloud into 2D "pancakes", whereas subjecting the atoms to two perpendicular optical lattices dices the cloud into an array of 1D "cigars" ([7] and e.g. experiment [8]), provided that the level separation in the optical lattice potential is much larger compared to the temperature, so that the atoms stay in the transverse ground state. By varying the number and location of the counter propagating laser pairs, lattices of different geometry can be obtained. The interatomic interaction strength is also tunable, for example for hyperfine-split alkali atoms with the help of Feschbach resonances [7, 9, 10].

Such versatility makes these systems not just excellent testing grounds for theoretical predictions, but also allows to use them to perform quantum simulations, i.e. to engineer model Hamiltonians relevant to various other branches of physics. For example, ultracold atomic gases confined to lattices are used to simulate model Hamiltonians relevant in theories of topological states of matter [11], artificial gauge fields [12], and let emulate and study the models proposed to explain the high–T_c superconductivity [9].

There are also good reasons to use the mentioned experimental capabilities to study 1D systems. One reason is that 1D systems are perfect for investigating strong interactions, as in 1D, qualitatively speaking, the atoms cannot move to avoid each other completely and the interactions, however weak, play the dominant role. There are certain reservations about studying 1D vs. 3D, which are especially relevant in the context of BEC and superfluidity. After all, it is known that phenomena of superfluidity and Bose–Einstein condensation are related [13, 14], however from the elementary theory of non-interacting Bose gas it is known that due to the phase space limitations the condensation is not even possible in 1D [15]. In more rigorous terms, Hohenberg [16] and Mermin and Wagner [17] have proven that the phase transition associated with the onset of superfluidity in 3D does not happen in lower dimensions.

It turns out, however, that a) interactions matter b) superfluidity, which is closely related to, but is not the same as BEC can be observed in 1D [18]. In addition, BEC in 1D can be achieved if one manages to change the available phase space volume by putting the atoms in a non homogeneous trap [19]. It also turns out that in 1D time- and length scales matter: there are regimes in 1D where the system behaves as BEC for long enough time, or is correlated on large enough length scales to be considered superfluid for all purposes.

The other important reason is simply because 1D is more analytically and numerically tractable than 3D. In the 70+ years between the prediction of Bose-Einstein condensation in 1924 [20, 21, 22] and its experimental achievement in 1995 an extensive body of theoretical results has been accumulated: Tizsa's and Landau's two-fluid hydrodynamics [23, 24, 25, 26], Bogoliubov theory [27] and Gross-Pitaevskii (GP) [28] equation to mention just a few, but many of them were mean-field approximations (Bogoliubov, GP), since the microscopic calculations are so notoriously difficult to deal with, especially in the presence of the BEC [18, 29, 30]. It is only in the area of 1D systems that several (remarkable) exact results were obtained.

The 1D Bose gas with point interactions was solved using Bethe ansatz by Lieb and Liniger [31, 32], later extended by Yang and Yang [33]. The Lieb-Liniger model has two branches of the spectrum, corresponding to particle-like and hole-like excitations: compare this to spectrum of free fermions in 1D and contrast with Bogoliubov's phonons [27]. The original work by Lieb and Liniger was extensively developed and generalized [34], but unfortunately scattering in 3D is not quite the same as scattering in 1D, and Bethe ansatz solution cannot be extended to higher dimensions. Results of Lieb and Liniger showed that the 1D interacting system is not that different from the 3D in its low–energy excitation spectrum and equation of state. A field-theoretic approach by Dyson and Beliaev [29, 30], and developed in full glory by Popov [18] also concurred: whereas no actual condensation is taking place, the system can behave as a BEC / superfluid on certain length scales, with the order parameter showing quasi–long range

order. With one branch of the Lieb-Liniger excitation spectrum being attributed to phonons, the second one was identified with the solitonic solutions of Gross-Pitaevskii equation [35, 36] (which were eventually observed in the 1D ultracold setting [37]).

The relative simplicity of 1D is also manifest at the other end of the spectrum: approximate solutions and effective models. A simple fact that in a single dimension two interacting particles cannot get one around another has lead to the idea that low-energy excitations in 1D systems are phonons which developed into Tomonaga-Luttinger Liquid model [38, 39], also known as linear hydrodynamics. It was originally conceived to describe 1D fermions, and is exact for fermions with linear unbounded spectrum [40], but also works for bosons [18], since the low-energy excitations for bosons also have linear dispersion (e.g. see Lieb-Liniger model). This also can be qualitatively understood from the Bose-Fermi correspondence. In 1D exchange statistics is not defined as well as in 3D, since the only way two particles can be "exchanged" is by colliding one with another. That way in 1D the Pauli exclusion principle between two (spinless) fermions is qualitatively the same as infinitely strong interaction between two bosons, in both cases the two particles cannot occupy the same spot. In 1D there is a strict mapping between weakly interacting fermions and strongly interacting bosons [41, 42], so if the Luttinger Liquid model is suitable to describing fermions, it should also be applicable to describing (interacting) bosons.

Qualitatively one may think of hydrodynamics as sampling the smooth field configurations, in a functional integral sense, and leaving out all the possible sharp ones. After the introduction of the Luttinger Liquid paradigm it was assumed that exact excited states could be obtained by including higher order terms into the effective hydrodynamic action, e.g. see the discussion of higher order corrections in the original paper by Haldane [38] and work by Andreev [43], which calculated the contribution of such corrections to certain correlation functions. In fact there is a recent work by Lamacraft and Kulkarni [44] that studies the dynamics of 1D bosons using quantum hydrodynamics, obtained from the Gross-Pitaevskii equation (i.e. already containing

higher order terms corresponding to non-linear dispersion). However, as will be shown in this Thesis, dynamical correlation functions depend on the detailed knowledge of the actual set of excited states of the system and counting only smooth configurations is not enough. In this respect it is helpful to think about the exact excitations of the Lieb-Liniger model that rather resemble particle / hole excitations of the free fermion gas than collective phonons. In some general sense the phonons of effective hydrodynamics should be expressed in terms of infinite combination of original "particles" and "holes",

$$\rho_k = \sum_p c_p^\dagger c_{k+p}, \tag{1.1}$$

and this "infinity" makes the non-hydrodynamic effects non-perturbative in hydrodynamic variables [45].

In the Chapter 2 of this Thesis we study the momentum-momentum correlator

$$\mathcal{G}(k,k') = \langle \delta n_k \delta n_{k'} \rangle \,, \tag{1.2}$$

a static correlation function, using various methods: Bogoliubov theory, theory of quasicondensate, and numerically using the classical field approach [46]. This is the type of correlation function that can be computed with the help of hydrodynamic description, which we employ to describe the 1D regime of quasicondensate. This problem is of interest because many-body correlation functions and their dependence on temperature, amount of disorder, and values of external (e.g. magnetic) fields contain a great deal of information about the many–particle systems. Recent advances in techniques of preparing and probing the ultracold atomic systems have led to considerable experimental effort in probing such many-body phenomena as the Hanbury Brown-Twiss effect [47, 48, 49, 50, 51, 52], higher-order coherences [53], phase fluctuations

in quasicondensates [54, 55], superfluid to Mott-insulator transition [56, 57, 58, 59], the phase diagram of the 1D Bose gas [60, 61] and others. However, all the experiments measuring 1D correlation functions mentioned probed either the equilibrium position–space or non equilibrium momentum–space correlators. That was the motivation why we decided to concentrate on the correlator (1.2). In Chapter 3 we use the numerics done for the classical field approximation treatment of Eq. (1.2) in Section 2.4 to derive the particle number distribution in a weakly interacting 1D gas.

In Chapter 4 we change focus and discuss at length non-hydrodynamic behavior in 1D systems, taking the Dynamical Structure Factor as example. We show that the hydrodynamic response function, even with higher-order corrections to the Luttinger Liquid model is not applicable to all frequency-momentum regimes, and derive the frequency threshold above which sharp non-hydrodynamic excitations of the system matter. Based on the numerical values of experimental parameters, we reach the conclusion that non-hydrodynamic behavior should be observable in dynamical experiments with weakly interacting bosons.

Chapter 2

Two–body momentum correlation functions

In our joint work with I. Bouchoule, K. V. Kheruntsyan and D. M. Gangardt [62] we have investigated the two-body momentum correlation function,

$$\mathcal{G}(k,k') = \langle \delta n_k \delta n_{k'} \rangle = \langle n_k n_{k'} \rangle - \langle n_k \rangle \langle n_{k'} \rangle , \tag{2.1}$$

where n_k is the occupation operator of the mode with momentum $\hbar k$ and $\delta n_k = n_k - \langle n_k \rangle$ is the fluctuation of the occupation number. 1D systems are particularly interesting setting to study the correlation functions because the long range order is not possible [16, 17], and depending on the ratio of the correlation length to the system size several qualitatively different regimes for the correlation functions are possible for the correlation functions, see Fig. 2.1. The motivation for this work was to investigate how the strong positive correlations between the opposite momenta,

$$\mathcal{G}(k,-k), \tag{2.2}$$

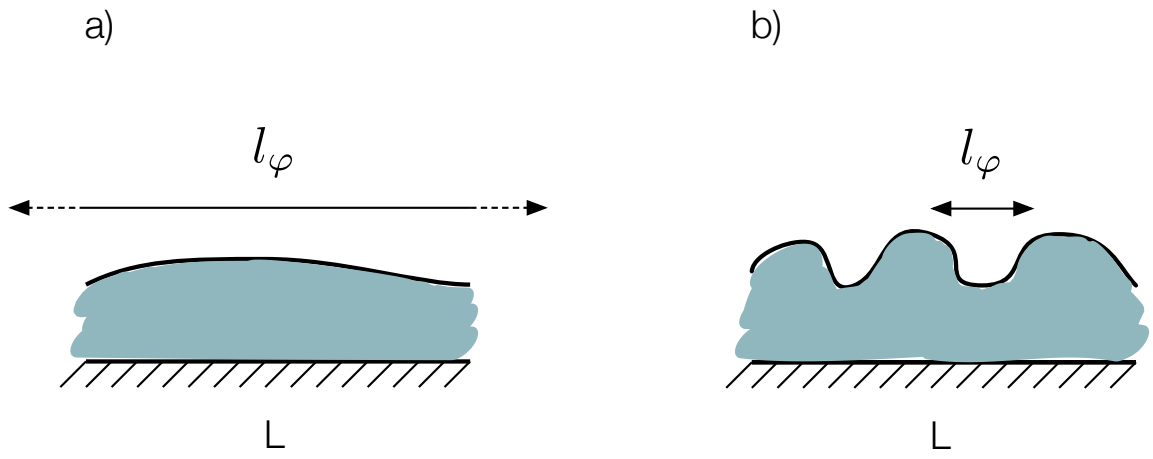


Figure 2.1: Phase of the $1D$ liquid in different regimes: a) Phase correlation length l_φ is larger than system size L, therefore this finite system can be treated as phase-coherent superfluid where in the regime of weak interactions Bogoliubov theory is applicable. For true condensate we expect not only ideal equal ($k' = k$), but also opposite ($k' = -k$) momenta correlations because of the nature of the Bogoliubov ground state, containing pairs of particles with equal and opposite momenta b) $l_\varphi \ll L$, phase fluctuates significantly over the length of the system. Our work [62] was dedicated to investigating what happens to the opposite-momenta correlations away from the true condensate regime.

expected from the Bogoliubov theory of the Bose-Einstein condensed gas are washed out by the phase fluctuations of the 1D quasicondensate and almost disappear when the long range order is destroyed. $\mathcal{G}(k, k')$ is a static (equilibrium) correlator, which makes it a fair game for the application of linear hydrodynamics. We see how the Luttinger liquid approach yields negative correlations for opposite momenta of small magnitude and use the Classical Field approximation to study how the Bogoliubov result crosses over to the quasicondensate regime as the ratio of the coherence length of the sample to its size (l_φ/L) decreases.

The outline of this Chapter is as follows: In Section 2.1 we set the notation and give general information about the behavior of the momentum–momentum correlator in 1D from the statistical physics point of view. In Section 2.2 we summarize the behavior expected in the presence of the true BEC. In Section 2.3 we derive results applicable to the case of the phase fluctuating quasicondensate. In Section 2.4 we use the Classical Field approximation to study how the two results cross over one into the other, and in Section 2.5 we discuss some experimentally relevant aspects.

This chapter is almost entirely based on the content of the previously published paper [62] where M. Arzamasovs has been one of the authors. My contribution was doing numerical work and writing up sections on the Classical Field Approximation, however for the sake of clarity I outline the content of the whole of the paper [62] in this Chapter.

2.1 Definitions and general properties

Let us consider a uniform gas of point interacting bosons, described by the Lieb-Liniger Hamiltonian [31], which when expressed in terms of second-quantized operators ψ, $\psi^\dagger$, takes the

following form:

$$H = \int_0^L \mathcal{H}\left(\psi, \psi^\dagger\right) dx = \int_0^L \left(-\frac{\hbar^2}{2m}\psi^\dagger \frac{\partial^2 \psi}{\partial r^2} + \frac{g}{2}\psi^\dagger\psi^\dagger\psi\psi - \mu\psi^\dagger\psi \right) dx, \tag{2.3}$$

where m is the bare mass of the Bosonic particles and $g > 0$ is the (repulsive) interaction constant. We work in the grand canonical ensemble with chemical potential μ, that fixes the average density $\rho_0(T, \mu) = \langle \psi^\dagger \psi \rangle$. We assume that the system has length L with periodic boundary conditions. We also restrict ourselves to weak inter particle interactions, which is expressed by the requirement that dimensionless interaction parameter γ is small, $\gamma = mg/\hbar^2\rho_0 \ll 1$ [31].

The momentum distribution $\langle n_k \rangle$ and its correlation function $\mathcal{G}(k, k')$ (see Eq. (2.1)) are related to the one and two-particle Bosonic field correlators,

$$G_1(x_1, x_2) = \left\langle \psi^\dagger(x_1)\psi(x_2) \right\rangle \tag{2.4}$$

and

$$G_2(x_1, x_2, x_3, x_4) = \left\langle \psi^\dagger(x_1)\psi(x_2)\psi^\dagger(x_3)\psi(x_4) \right\rangle, \tag{2.5}$$

via the Fourier transforms

$$\langle n_k \rangle = \left\langle \psi_k^\dagger \psi_k \right\rangle = \frac{1}{L}\int_0^L G_1(x_1, x_2) e^{ik(x_1 - x_2)}\, d^2x \tag{2.6}$$

and

$$\mathcal{G}(k, k') = \frac{1}{L^2}\int_0^L d^4x\, e^{ik(x_1 - x_2)} e^{ik'(x_3 - x_4)} \left(G_2(x_1, x_2, x_3, x_4) - G_1(x_1, x_2) G_1(x_3, x_4) \right), \tag{2.7}$$

where d^2x and d^4x stand for $dx_1 dx_2$ and $dx_1 dx_2 dx_3 dx_4$, respectively.

Several properties of the correlation function $\mathcal{G}(k,k')$ follow from general statistical mechanics and symmetry arguments. First of all, $\mathcal{G}(k,k')$ obeys the following sum rule:

$$\sum_{k,k'} \mathcal{G}(k,k') = \left\langle N^2 \right\rangle - \left\langle N \right\rangle^2 , \tag{2.8}$$

where

$$N = \int_0^L dx\, \psi^\dagger(x)\psi(x) \tag{2.9}$$

is the operator of the total number of particles. Equation (2.8) can be verified starting directly from the definition Eq. (2.7) and using the following summation formula:

$$\sum_{k=-\infty}^{\infty} e^{ik(x_1-x_2)} = L\delta(x_1-x_2). \tag{2.10}$$

This allows for the explicit summation of $\mathcal{G}(k,k')$ over both arguments

$$\begin{aligned} \textstyle\sum_{k,k'} \mathcal{G}(k,k') &= \textstyle\int_0^L d^4x\, \delta(x_1-x_2)\delta(x_3-x_4)\left(G_2(x_1,x_2,x_3,x_4) - G_1(x_1,x_2)G_1(x_3,x_4)\right) \\ &= \textstyle\int_0^L d^2x\, \left(G_2(x_1,x_1,x_3,x_3) - G_1(x_1,x_1)G_1(x_3,x_3)\right) = \left\langle N^2 \right\rangle - \left\langle N \right\rangle^2 . \end{aligned} \tag{2.11}$$

The sum rule Eq. (2.8) can be developed further by applying the fluctuation-dissipation relation. In the grand canonical ensemble the right-hand side of Eq. (2.8) is given by the following thermodynamic identity:

$$\left\langle N^2 \right\rangle - \left\langle N \right\rangle^2 = k_B T \frac{\partial \left\langle N \right\rangle}{\partial \mu} = k_B T L \frac{\partial \rho}{\partial \mu} \tag{2.12}$$

which follows directly from the definitions of $\langle N \rangle$ and $\langle N^2 \rangle$ [63].

Another set of properties is related to the homogeneity and periodicity of the system Eq.

(2.3), which result in symmetries of space–domain correlation functions with respect to interchange of their variables and simultaneous change of sign of all the arguments. For example, in Eq. (2.4) one could as well write $G_1(|x_1 - x_2|)$. These symmetries manifest themselves in the fact that $\mathcal{G}(k, k')$ is symmetric around the $k' = k$ and $k' = -k$ axes (see Fig. 2.2).

Indeed, in the first case, interchanging k' and k places in $\mathcal{G}(k, k')$ and relabeling

$$(x_1, x_2) \leftrightarrow (x_3, x_4) \tag{2.13}$$

gives

$$\mathcal{G}(k, k') - \mathcal{G}(k', k) = \frac{1}{L^2} \int d^4x \, e^{ik(x_1 - x_2)} e^{ik(x_3 - x_4)} \left(G_2(x_1, x_2, x_3, x_4) - G_2(x_3, x_4, x_1, x_2) \right). \tag{2.14}$$

Applying Bosonic commutation relations to the difference of the two-particle correlators

$$G_2(x_1, x_2, x_3, x_4) - G_2(x_3, x_4, x_1, x_2) = G_1(x_1 - x_4)\delta(x_2 - x_3) - G_1(x_3 - x_2)\delta(x_1 - x_4) \tag{2.15}$$

and integrating out the δ-functions gives

$$\begin{aligned} \mathcal{G}(k, k') - \mathcal{G}(k', k) \;=\; & \frac{1}{L^2} \int dx_1 dx_2 dy \, e^{ik(x_1 - x_2)} \left(e^{ik'(x_2 - y)} G_1(x_1 - y) \right) - \\ & - \frac{1}{L^2} \int dx_1 dx_2 dy \, e^{ik(x_1 - x_2)} \left(e^{ik'(y - x_1)} G_1(y - x_2) \right) \end{aligned} \tag{2.16}$$

where we replace either x_4 or x_3 by y. That the two integrals on the right hand side in Eq. (2.16) are the same can be understood by replacing e.g. $x_2 - y \to x_2$ etc. and integrating out

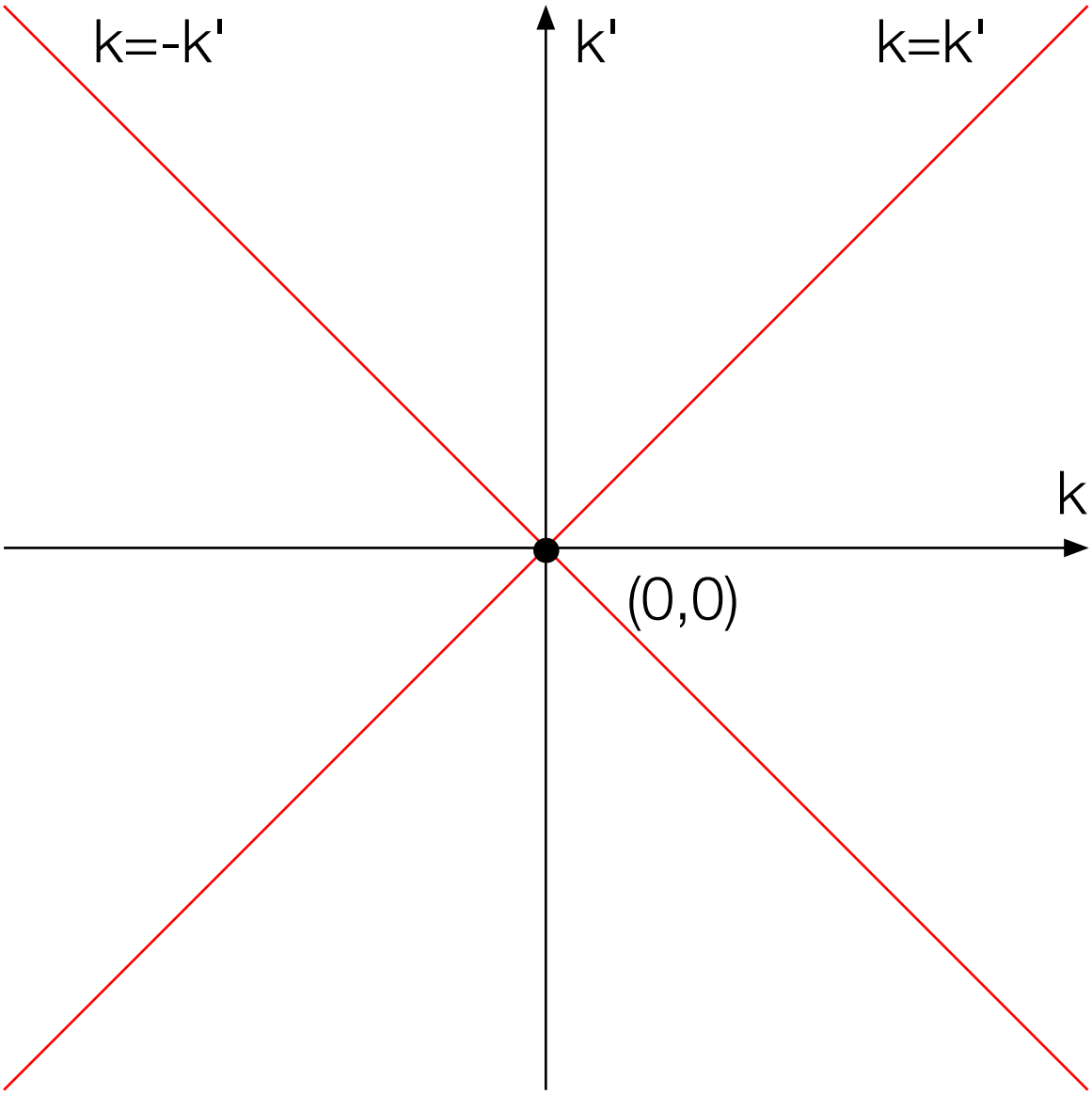


Figure 2.2: $\mathcal{G}(k,k')$ (and $\tilde{\mathcal{G}}(k,k')$) are symmetric with respect to the (red) lines $k'=k$ and $k'=-k$ on the (k,k') plane, which is the manifestation of spatial homogeneity of the system.

y (periodicity),

$$\begin{aligned}\mathcal{G}(k,k') - \mathcal{G}(k',k) &= \frac{1}{L}\int dx_1 dx_2\, e^{ik(x_1-x_2)}\left(e^{ik'(x_2)}G_1(x_1)\right) - \\ &\quad -\frac{1}{L}\int dx_1 dx_2\, e^{ik(x_1-x_2)}\left(e^{ik'(-x_1)}G_1(-x_2)\right),\end{aligned} \tag{2.17}$$

and then making the replacement $-x_2 \to x_1$, $-x_1 \to x_2$ in the second integral.

Proving that $\mathcal{G}(k,k') = \mathcal{G}(-k',-k)$ is done similarly. First of all we use the fact that $\mathcal{G}(k,k') = \mathcal{G}(k',k)$. Then symmetry $\mathcal{G}(k',k) = \mathcal{G}(-k',-k)$ may be understood from the fact, that under simultaneous flipping of the signs of the coordinates, $x_i \to -x_i$, the position–space correlators must not change, since the choice of the positive x direction is arbitrary.

Finally, another statement can be made when the one–particle correlator $G_1(x_1 - x_2)$ has a finite correlation length $l_\varphi \ll L$. In that case $\mathcal{G}(k,k')$ can be split into "singular" and "regular" parts. Indeed, assume that $G_1(x)$ has the following form:

$$G_1(x) \propto e^{-|x|/l_\varphi}, \tag{2.18}$$

for large x. Then the two–body correlation function Eq. (2.5) has the following asymptotic limits:

$$G_2(x_1,x_2,x_3,x_4) \simeq G_1(x_1-x_2)G_1(x_3-x_4), \tag{2.19}$$

$$|x_1 - x_3| \gg l_\varphi;\ |x_1-x_2|, |x_3-x_4| \lesssim l_\varphi$$

and

$$G_2(x_1,x_2,x_3,x_4) \simeq G_1(x_1-x_4)G_1(x_2-x_3) + G_1(x_1-x_4)\delta(x_2-x_3), \tag{2.20}$$

$$|x_1 - x_2| \gg l_\varphi;\ |x_1-x_4|, |x_2-x_3| \lesssim l_\varphi.$$

The δ-function in Eq. (2.20) appears as a result of normal ordering of operators in Eq. (2.5). By including the asymptotic limits Eqs. (2.19), (2.20) into the definition of G_2, we write

$$\begin{aligned} G_2(x_1,x_2,x_3,x_4) &= \\ &= G_1(x_1-x_2)G_1(x_3-x_4)+G_1(x_1-x_4)G_1(x_2-x_3)+ \\ &\quad +G_1(x_1-x_4)\delta(x_2-x_3)+\tilde{G}_2(x_1,x_2,x_3,x_4), \end{aligned} \tag{2.21}$$

where $\tilde{G}_2(x_1,x_2,x_3,x_4)$ is simply defined as the difference of $G_2(x_1,x_2,x_3,x_4)$ and the asymptotic limits Eqs. (2.19) and (2.20). By then substituting Eq. (2.21) into Eq. (2.7) we obtain

$$\mathcal{G}(k,k') = \left(\langle n_k\rangle + \langle n_k\rangle^2\right)\delta_{k,k'} + \tilde{\mathcal{G}}(k,k'), \tag{2.22}$$

which explicitly splits $\mathcal{G}(k,k')$ into a "singular" part, proportional to Kroneker delta, which in the continuous limit becomes delta–function $\delta(k-k')$, and a regular part $\tilde{\mathcal{G}}(k,k')$. Notice, that for noninteracting bosons Wick's theorem can be applied to the average Eq. (2.5) to find that $\tilde{\mathcal{G}}(k,k')$ vanishes and the correlation function becomes trivial, $\mathcal{G}(k,k') \propto \delta_{k,k'}$. This implies that $\tilde{\mathcal{G}}(k,k')$ accounts for the effect of elastic collisions on the momentum distribution.

2.2 $\mathcal{G}(k,k')$ in the true condensate

In this Section we summarize the expected behavior of $\mathcal{G}(k,k')$ in the presence of the true Bose-Einstein condensate using the Bogoliubov theory.

At $T=0$ the one-particle correlation function $G_1(x_1-x_2)$ decays algebraically[18, 40, 64] as

$$G_1(x_1,x_2) \approx \left(\xi/\left|x_1-x_2\right|\right)^{\sqrt{\gamma}/2\pi}, \qquad |x_1-x_2| \gg \xi, \tag{2.23}$$

where ξ is the healing length,

$$\xi = \frac{\hbar}{\sqrt{mg\rho}}, \tag{2.24}$$

and γ parameterizes the interaction strength,

$$\gamma = \frac{mg}{\hbar^2\rho_0}. \tag{2.25}$$

In the weakly interacting regime, $\gamma \ll 1$, this leads to a very slow decay of correlations with an exponentially large phase correlation length. This can be seen by estimating the distance at which the correlation function Eq. (2.23) decreases by a factor of e. We take the logarithm of Eq. (2.23):

$$\frac{\sqrt{\gamma}}{2\pi} \ln\left(\frac{x + l_\varphi}{x}\right) = 1 \qquad \Rightarrow \qquad \frac{l_\varphi}{x} = e^{2\pi/\sqrt{\gamma}} - 1 \approx e^{2\pi/\sqrt{\gamma}} \tag{2.26}$$

where the final step is justified due to smallness of γ. By setting $x = \xi$ in Eq. (2.26) to set the scale we get the lower bound on the phase correlation length

$$l_\varphi^{(0)} = \xi e^{2\pi/\sqrt{\gamma}}. \tag{2.27}$$

For a detailed investigation of the one-dimensional correlation function G_1 see [65].

The algebraic decay of $G_1(x)$, Eq. (2.23), also holds for finite temperatures, but only up to distances of the order of thermal wavelength of the phonons [64, 40]

$$l_T = \frac{\hbar^2}{mk_BT\xi} = \frac{\hbar}{k_BT}\sqrt{\frac{g\rho}{m}}, \tag{2.28}$$

which is a manifestation of the physical fact that at $T > 0$ the long wavelength (low energy) phonons are excited, with l_T being the minimum wavelength of excitations at temperature T.

The thermal phonons destroy the coherence at large length scales and for distances $x \gg l_T$ one sees the exponential decay of correlations with the characteristic phase coherence length

$$l_\varphi(T) = \frac{\hbar^2 \rho}{m k_B T}. \tag{2.29}$$

Although it is known that in one dimension Bose-Einstein condensation does not occur[1] [15, 16], if we now restrict ourselves to the system of size $L \ll \min\left(l_\varphi^{(0)}, l_\varphi\right)$ it is still possible to apply the notion of the long-range order and treat it as Bose-Einstein condensed. For such systems the momentum correlations have been investigated using the Bogoliubov theory [66, 67] and it turns out that the correlation function $\mathcal{G}(k, k')$ is non–zero only on the lines $k' = k$ and $k' = -k$.

For $k' = k$ one finds

$$\mathcal{G}(k, k) = \langle n_k \rangle + \langle n_k \rangle^2, \tag{2.30}$$

which resembles the result for the non–interacting Bose gas (see end of Section 2.1), with the difference that $\langle n_k \rangle$ is not the Bose occupation number

$$\langle n_k \rangle = \frac{1}{\exp\left(\left(E_k - \mu\right)/k_B T\right) - 1}, \tag{2.31}$$

but rather

$$\langle n_k \rangle = \left(1 + 2\tilde{n}_k\right) \frac{E_k + g\rho}{2\epsilon_k} - \frac{1}{2} \tag{2.32}$$

where

$$E_k = \frac{\hbar^2 k^2}{2m}, \qquad \epsilon_k = \sqrt{E_k \left(E_k + 2g\rho\right)} \tag{2.33}$$

[1] Condensation in 1D is possible in the presence of a trap [19], however in our discussion we always assume trapping potential to be small and insignificant when discussing correlation functions of the system.

are free-particle and Bogoliubov dispersions, respectively, and

$$\tilde{n}_k = \frac{1}{\exp\left(\epsilon_k / k_B T\right) - 1} \tag{2.34}$$

is the Bose occupation number of the Bogoliubov modes.

For the opposite momenta the correlator is

$$\mathcal{G}(k,-k) = \left(1 + 2\tilde{n}_k\right)^2 \left(\frac{g\rho}{2\epsilon_k}\right)^2. \tag{2.35}$$

At this point it is convenient to introduce the function

$$\mathcal{P}(k) = \frac{\mathcal{G}(k,-k)}{\mathcal{G}(k,k)} \tag{2.36}$$

that characterizes the relative importance of opposite and equal momenta correlations. $\mathcal{P}(k) = 1$ means perfect correlation between momenta k and $-k$, whereas $\mathcal{P}(k) = 0$ means no correlation at all.

Based on Eqs. (2.30)-(2.35) the Bogoliubov theory predicts the following behavior for $\mathcal{P}(k)$, see Fig. 2.3. At $T = 0$, $\tilde{n}_k = 0$ for all k, and

$$\frac{\mathcal{G}(k,-k)}{\mathcal{G}(k,k)} = \frac{\left(g\rho\right)^2}{2\epsilon_k\left(E_k + g\rho - \epsilon_k\right) + \left(E_k + g\rho - \epsilon_k\right)^2} = \frac{\left(g\rho\right)^2}{\left(g\rho\right)^2} = 1, \tag{2.37}$$

which means perfect correlation at all momenta. This can be understood from a nature of Bogoliubov vacuum, which is the non-interacting condensate depleted by the pairs of particles having equal in magnitude but opposite momenta.

At $T > 0$ we can distinguish three regimes. For $k \ll 1/\xi$ the Bogoliubov dispersion $\epsilon_k \ll g\rho$

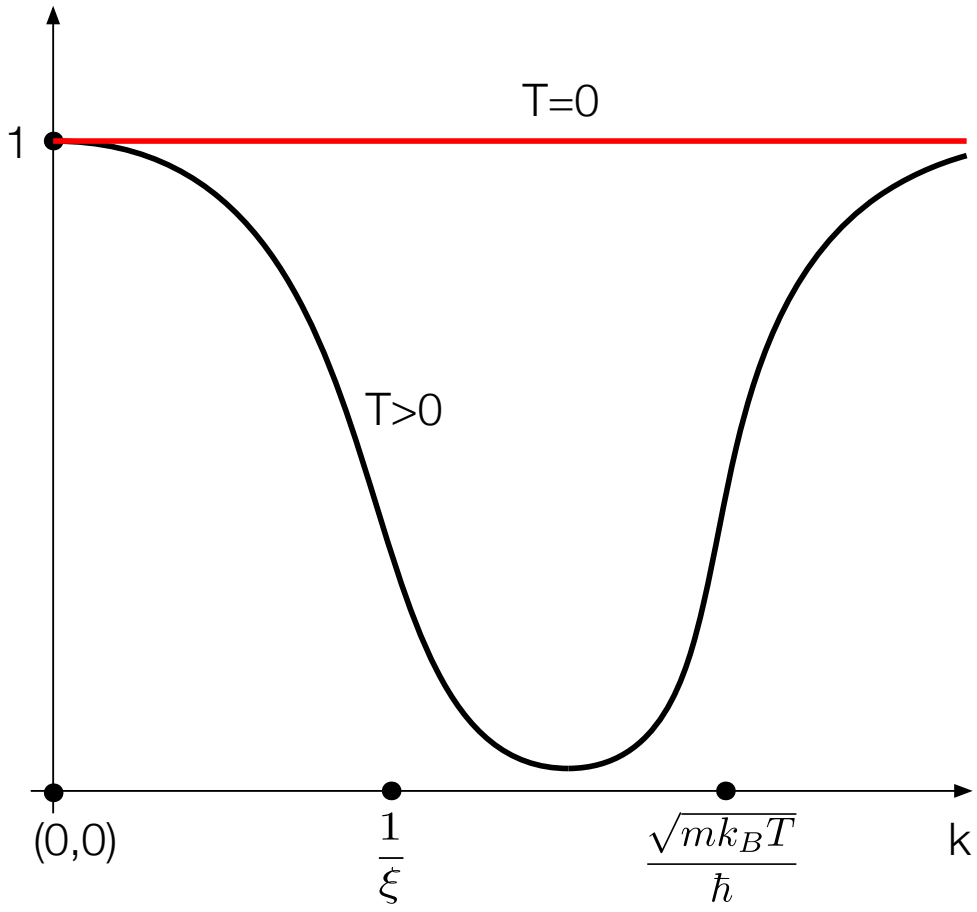


Figure 2.3: $\mathcal{P}(k)$ in Bogoliubov theory. The black line shows different regimes at $T > 0$, whereas the red line corresponds to a constant value of $\mathcal{P}(k) = 1$ at $T = 0$.

and therefore

$$\mathcal{G}(k,k) = \langle n_k \rangle + \langle n_k \rangle^2 \approx \langle n_k \rangle^2 \approx (1+2\tilde{n}_k)^2 \left(\frac{E_k + g\rho}{2\epsilon_k} \right)^2 \approx \mathcal{G}(k,-k), \tag{2.38}$$

which leads to

$$\mathcal{P}(k) \approx 1, \qquad k \ll \frac{1}{\xi}. \tag{2.39}$$

In the regime $1/\xi < k < \sqrt{mk_BT}/\hbar$, $\mathcal{P}(k)$ decreases rapidly,

$$\mathcal{P}(k) \ll 1, \qquad \frac{1}{\xi} < k < \frac{\sqrt{mk_BT}}{\hbar}. \tag{2.40}$$

Finally, for $k > \sqrt{mk_BT}/\hbar$ we enter the effective zero-temperature regime and recover the result of Eq. (2.37),

$$\mathcal{P}(k) \approx 1, \qquad k \gg \frac{\sqrt{mk_BT}}{\hbar}. \tag{2.41}$$

In the next Section we will show how result Eq. (2.39) changes when the sample size exceeds its correlation length.

2.3 Quasicondensate

The results obtained in Section 2.2 are valid when the phase fluctuations are not relevant, i.e. when the system size is much smaller than the phase correlation length, $L \ll l_\varphi$. This condition requires very low temperatures, and is in general difficult to satisfy for the 1D gases. Therefore in large enough 1D systems or at high enough temperatures the long range order is destroyed by phase fluctuations with temperature-dependent correlation length l_φ (Eq. (2.29)). For large system size, $l_\varphi \ll L$, the system enters the quasicondensate regime [65], in which the density fluctuations are still suppressed (i.e. density $\psi^\dagger\psi$ is correlated at large distances) while the

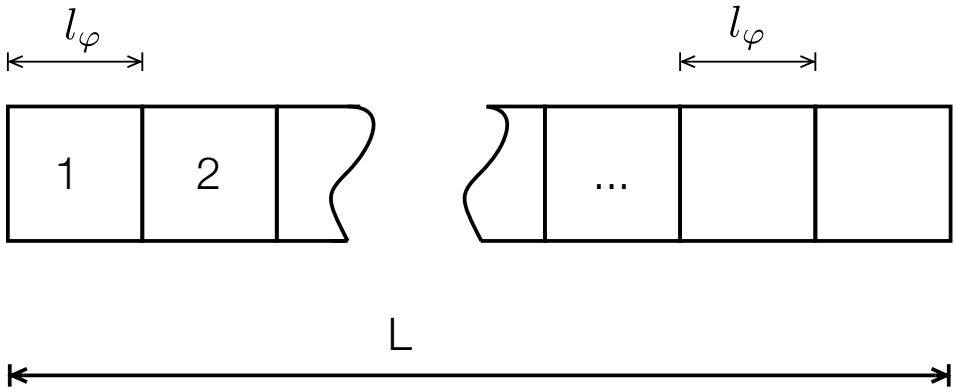


Figure 2.4: Illustration to the simple model of the quasicondesate regime. The sample is assumed to consist of many mutually phase-uncorrelated "condensate" regions of length $l_\varphi = l_\varphi(T)$, Eq. 2.29.

phase does fluctuate. In this Section we show that due to such fluctuations of the phase the two–body momentum correlation function between opposite momenta is expected to behave as

$$\mathcal{G}(k,-k) \propto \frac{l_\varphi}{L}, \tag{2.42}$$

and therefore to vanish in the large L (thermodynamic) limit.

We first give a simple yet insightful picture of physics of the quasicondensate regime, put forward by I. Bouchoule. We divide our system, now satisfying the condition $l_\varphi \ll L$, into many segments of size l_φ each labelled by $\alpha = 1, 2, \ldots, [L/l_\varphi]$ (see Fig. (2.4)). We then assume that the phase stays uniform across any single segment, but phases on the two different segments are completely uncorrelated. Moreover, we assume that to each of the small segments we can apply Bogoliubov theory, as if it was a true condensate. Under these assumptions we can write down the position-space annihilation operators, corresponding to each segment, as follows:

$$\psi_\alpha(x) = e^{i\varphi_\alpha}\left(\sqrt{\rho} + \frac{1}{\sqrt{l_\varphi}}\sum_{k\neq 0}\delta\psi_{\alpha,k}e^{ikx}\right). \tag{2.43}$$

Here ρ is the average density, φ_α is the phase of the α-th segment and $\delta\psi_{\alpha,k}$ are the plane wave modes quantized in units of $2\pi/l_\varphi$.

By writing the momentum component of the full field

$$\psi_k = \frac{1}{\sqrt{L}} \int_0^L dx\, \psi(x) e^{-ikx}, \qquad k \neq 0 \tag{2.44}$$

in terms of the field operators $\psi_\alpha(x)$

$$\psi_k = \frac{1}{\sqrt{L}} \left[\int_0^{l_\varphi} dx\, \psi_1(x) e^{-ikx} + \int_{l_\varphi}^{2l_\varphi} dx\, \psi_2(x) e^{-ikx} + \ldots + \int_{(\alpha-1)l_\varphi}^{\alpha l_\varphi} dx\, \psi_\alpha(x) e^{-ikx} + \ldots \right] \tag{2.45}$$

we obtain

$$\psi_k = \sqrt{\frac{l_\varphi}{L}} \sum_\alpha \delta\psi_{\alpha,k} e^{i\varphi_\alpha} \tag{2.46}$$

and hence the following expression for the momentum correlation function

$$\langle n_k n_{k'} \rangle = \left(\frac{l_\varphi}{L} \right)^2 \sum_{\alpha\beta\gamma\delta} \left\langle \delta\psi^\dagger_{\alpha,k} \delta\psi_{\beta,k} \delta\psi^\dagger_{\gamma,k'} \delta\psi_{\delta,k'} \right\rangle \overline{e^{-i(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)}}, \qquad k, k' \neq 0 \tag{2.47}$$

where the bar over the exponential stands for averaging over the random phases of different domains. Since each of the segments can be treated within the Bogoliubov theory, which is quadratic in $\delta\psi_{\alpha,k}$ and $\delta\psi^\dagger_{\alpha,k}$, we can use Wick's theorem to evaluate the four-operator average in Eq. (2.47). Only pairs belonging to the same segment can have non–zero average, since phases of different segments are uncorrelated

$$\overline{e^{-i(\varphi_\alpha - \varphi_\beta)}} = \delta_{\alpha,\beta}. \tag{2.48}$$

Among such pairs, only $\left\langle \delta\psi^{\dagger}_{\alpha,k}\delta\psi_{\alpha,k}\right\rangle$, $\langle\delta\psi_{\alpha,k}\delta\psi_{\alpha,-k}\rangle$ and $\left\langle \delta\psi^{\dagger}_{\alpha,k}\delta\psi^{\dagger}_{\alpha,-k}\right\rangle$ survive. Thus in this simple quasicondensate treatment we can only have correlations between the same and the directly opposite momenta, for all other combinations of k and k' in Eq. (2.47) the average $\langle n_k n_{k'}\rangle$ reduces to the the product of averages, $\langle n_k\rangle\langle n_{k'}\rangle$, thus rendering the correlator $\langle n_k n_{k'}\rangle - \langle n_k\rangle\langle n_{k'}\rangle$ zero.

Further, it can be verified from an equation similar to Eq. (2.47),

$$\langle n_k\rangle = \left(\frac{l_\varphi}{L}\right)\sum_{\alpha\beta}\left\langle \delta\psi^{\dagger}_{\alpha,k}\delta\psi_{\beta,k}\right\rangle\overline{e^{-i\left(\varphi_\alpha-\varphi_\beta\right)}} = \left(\frac{l_\varphi}{L}\right)\sum_{\alpha\beta}\left\langle \delta\psi^{\dagger}_{\alpha,k}\delta\psi_{\beta,k}\right\rangle\delta_{\alpha,\beta} = \left(\frac{l_\varphi}{L}\right)\sum_{\alpha}\left\langle \delta\psi^{\dagger}_{\alpha,k}\delta\psi_{\alpha,k}\right\rangle, \tag{2.49}$$

$\left\langle \delta\psi^{\dagger}_{\alpha,k}\delta\psi_{\alpha,k}\right\rangle$ is simply given by the Bogoliubov occupation number $\langle n_k\rangle$ of k-th mode Eq. (2.32). Either from the Classical Field approximation arguments or, equally, assuming large population of the modes under consideration, $\langle n_k\rangle \gg 1$, it is possible to show that

$$\langle\delta\psi_{\alpha,k}\delta\psi_{\alpha,-k}\rangle = \left\langle \delta\psi^{\dagger}_{\alpha,k}\delta\psi^{\dagger}_{\alpha,-k}\right\rangle \approx -\langle n_k\rangle\,. \tag{2.50}$$

Thus, according to Eqs. (2.1) and (2.22), the regular part of the momentum–momentum correlator is given by

$$\tilde{\mathcal{G}}(k,k') \approx \delta_{k,-k'}\left(\frac{l_\varphi}{L}\right)^2\sum_{\alpha\beta}\overline{e^{-i\left(\varphi_\alpha-\varphi_\beta\right)}}\,\langle n_k\rangle^2 = \delta_{k,-k'}\left(\frac{l_\varphi}{L}\right)\langle n_k\rangle^2\,. \tag{2.51}$$

In Eq. (2.51) the summation over β returns 1 (because of Eq. (2.48)) and the subsequent summation over α returns the number of segments, L/l_φ, hence the reduction in the power of (l_φ/L).

Thus in this simple picture, taking into account Eq. (2.51), we find that for equal momenta

$\tilde{\mathcal{G}}(k,k) = 0$, i.e. the momentum–momentum correlator is given by just the singular part

$$\mathcal{G}(k,k) = \langle n_k \rangle^2 + \langle n_k \rangle \approx \langle n_k \rangle^2 \tag{2.52}$$

for $\langle n_k \rangle \gg 1$, which holds for $k \ll 1/\xi$. For the opposite momenta

$$\mathcal{G}(k,-k) \approx \left(\frac{l_\varphi}{L}\right) \langle n_k \rangle^2 , \tag{2.53}$$

which means that the correlations between the opposite momenta are inversely proportional to the system size,

$$\mathcal{P}(k) \approx \frac{l_\varphi}{L} \ll 1, \qquad k \ll \frac{1}{\xi}, \tag{2.54}$$

i.e. are vanishingly small for $L \gg l_\varphi$. It should be pointed out that only the fluctuations of the phase of wavelength $\sim l_\varphi$ were considered, therefore result Eq. (2.54) can only be valid for $k > 1/l_\varphi$. To access all the values of k in the quasicondensate regime we will now use more rigorous Luttinger Liquid formalism.

The condition for the system to be in the quasicondensate regime is that its temperature is lower than the cross-over temperature [68],

$$T \ll T_{co} = \sqrt{\gamma}\frac{\hbar^2\rho^2}{2mk_B}. \tag{2.55}$$

In this regime the thermodynamics of the system is well described by the long wavelength, low energy phonons. To a certain degree of accuracy [69, 70], one can represent the "complex" operator $\psi(x)$ in terms of its "modulus" and "argument" $\sqrt{\rho(x)}$ and $\varphi(x)$,

$$\psi(x) \to \sqrt{\rho(x)}e^{-i\varphi(x)}, \tag{2.56}$$

$$\psi^{\dagger}(x) \rightarrow \sqrt{\rho(x)} e^{i\varphi(x)}, \tag{2.57}$$

and instead of the actual Hamiltonian, Eq. (2.3), one can describe the system by an effective Hamiltonian density,

$$\mathcal{H}_{LL} = \frac{\hbar^2 \rho}{2m} \left(\partial_x \varphi\right)^2 + \frac{g}{2} \left(\delta\rho\right)^2 . \tag{2.58}$$

In Eq. (2.58) $\delta\rho = \rho(x) - \rho_0$, $\delta\rho \ll \rho_0$, and the phase operator $\varphi(x)$ is the canonical conjugate of $\delta\rho(x)$, satisfying the following commutation relation:

$$[\delta\rho(x), \varphi(x')] = i\delta(x - x'). \tag{2.59}$$

The effective phononic theory with Hamiltonian density Eq. (2.58) is called linear hydrodynamics, or Luttinger Liquid model. It can be rigorously obtained from Eq. (2.3) by a procedure called bosonization [18, 39, 40].

In the quasicondensate regime, Eq. (2.55), and if the distances considered are much larger than the healing length ξ, the density fluctuations can be neglected in comparison to the fluctuations of the phase [40, 64, 71, 72]. In this manner, the one–particle correlator Eq. (2.4) becomes

$$G_1 \left(x_1, x_2\right) = \rho_0 \left\langle e^{i(\varphi(x_1) - \varphi(x_2))} \right\rangle . \tag{2.60}$$

Since the effective Hamiltonian Eq. (2.58) is quadratic in $\varphi(x)$, the average of the exponent in Eq. (2.60) can be written as the exponent of the average, using Wick's theorem, to give

$$G_1 \left(x_1, x_2\right) = \rho_0 e^{-\frac{1}{2}\left\langle (\varphi(x_1) - \varphi(x_2))^2 \right\rangle}. \tag{2.61}$$

Neglecting the contribution of quantum fluctuations and taking into account only thermal ones, evaluation of the mean–square phase fluctuation results in an exponential decay of the

one–particle correlator [40, 64, 72]

$$G_1(x_1, x_2) = \rho_0 e^{-|x_1 - x_2|/2l_\varphi}, \qquad |x_1 - x_2| \gg \xi \tag{2.62}$$

with l_φ given by Eq. (2.29), which in turn leads to the Lorenzian distribution of the momentum mode occupation numbers,

$$\langle n_k \rangle = \frac{4\rho_0 l_\varphi}{1 + (2l_\varphi k)^2}, \qquad k \ll 1/\xi. \tag{2.63}$$

Due to the same reason of quadraticity of Eq. (2.58) (and again neglecting the density fluctuations), the two–particle correlation function Eq. (2.5)

$$G_2(x_1, x_2, x_3, x_4) = \rho_0^2 \left\langle e^{i(\varphi(x_1) - \varphi(x_2) + \varphi(x_3) - \varphi(x_4))} \right\rangle \tag{2.64}$$

can also be re-expressed as the exponential of the average,

$$G_2(x_1, x_2, x_3, x_4) = G_2(x_1, x_2, x_3, x_4) = \rho_0^2 e^{-\frac{1}{2}\left\langle (\varphi(x_1) - \varphi(x_2) + \varphi(x_3) - \varphi(x_4))^2 \right\rangle}, \tag{2.65}$$

which in turn can be expressed entirely in terms of the one–particle correlators Eqs. (2.4), (2.62),

$$G_2(x_1, x_2, x_3, x_4) = \frac{G_1(x_1 - x_2) G_1(x_3 - x_4) G_1(x_1 - x_4) G_1(x_2 - x_3)}{G_1(x_1 - x_3) G_1(x_2 - x_4)}. \tag{2.66}$$

After Fourier transforming results Eqs. (2.62) and (2.66), the momentum–momentum correlator $\mathcal{G}(k, k')$ and its regular part $\tilde{\mathcal{G}}(k, k')$ can be computed. The regular part takes the form

$$\tilde{\mathcal{G}}(k, k') = \frac{l_\varphi}{L} (\rho_0 l_\varphi)^2 \mathcal{F}(2l_\varphi k, 2l_\varphi k'), \tag{2.67}$$

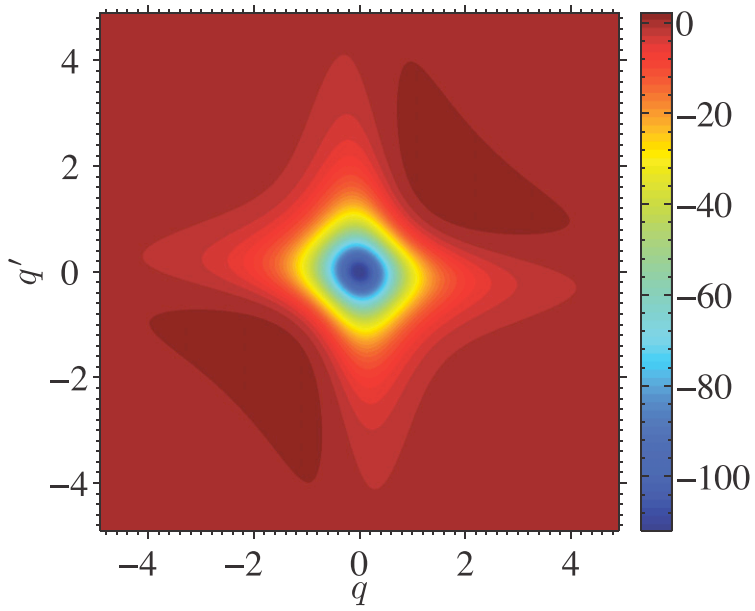
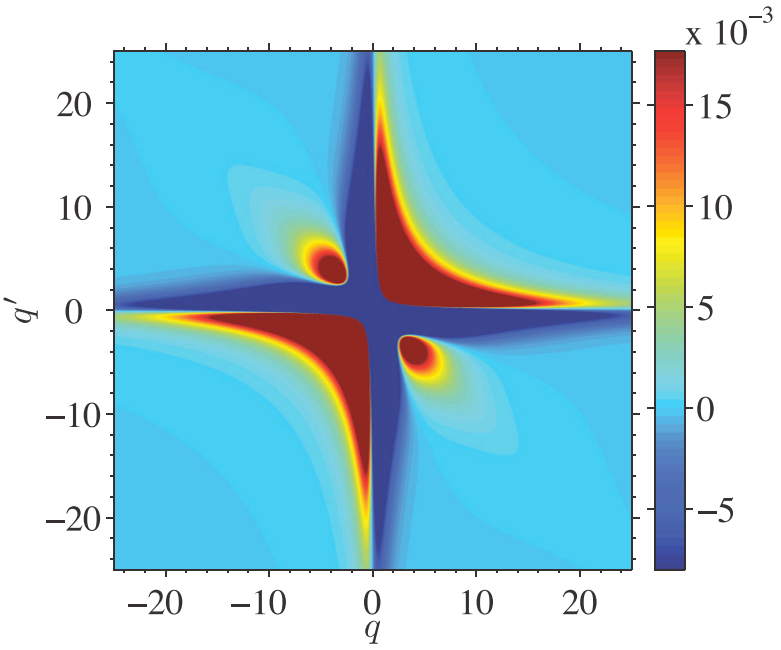


Figure 2.5: Regular part of the universal two-body momentum correlation function $\mathcal{F}(q, q')$, Eq. (2.68), in the quasicondensate regime.

Figure 2.6: Universal correlation function $\mathcal{F}(q, q')$, same as Fig. 2.5, but notice the different scales of values and q, q'.



where $\mathcal{F}(q, q')$ is the universal dimensionless function,

$$\mathcal{F}(q, q') = \frac{256\left[\left(q^2 + 3qq' + q'^2\right)qq' - 2\left(q^2 - qq' + q'^2\right) - 7\right]}{\left(q^2+1\right)^2\left(q'^2+1\right)^2\left[\left(q+q'\right)^2 + 16\right]}, \tag{2.68}$$

that essentially contains all the information about momentum-momentum correlations in the quasicondensate regime.

Eqs. (2.67) and (2.68) are among the key results of this Chapter. The first thing to notice in Eq. (2.67) is that scaling with the inverse system size that was qualitatively obtained in Eq. (2.54) is confirmed. This means, in particular, that perfect correlation between opposite momenta obtained in the Bogoliubov theory, Eq. (2.39) does not hold for the phase-fluctuating quasicondensate $L \gg l_\varphi$. Another consequence of the phase fluctuations is the broadening of the peaks at $k' = k$ and $k' = -k$: instead of two delta functions centered at these lines we

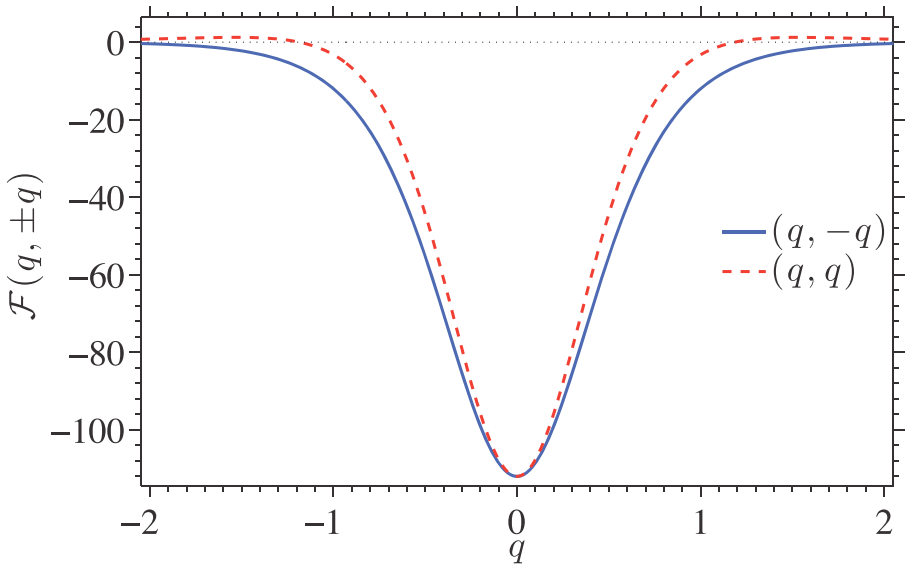


Figure 2.7: Universal correlation function $\mathcal{F}(q, q')$ evaluated for same ($q' = q$) and opposite ($q' = -q$) values of momenta.

get the correlator that is non–zero everywhere (see Eq. (2.68)) On the Figs. 2.5 and 2.6 one can study the behavior of $\mathcal{F}(q, q')$ at different scales. Because in the quasicondensate both two-particle and one-particle correlators Eqs. (2.62) and (2.66) depend on a single length scale l_φ, the correlation function $\mathcal{F}(q, q')$ decays on the length scale of 1, see e.g. Fig. 2.7. Also notice the appearance of regions with negative correlations on Figs. 2.5 and 2.6.

2.4 Classical field approach

We have seen that in the quasicondensate regime, $T \ll T_{co}$, the correlation function G_2 can be expressed in terms of G_1 and consequently $\mathcal{G}(k, k')$ contains all the same information as the momentum distribution n_k, Eq. (2.6). This, however, ceases to be true when the temperature approaches the crossover temperature T_{co} and the fluctuations of density become as important as the fluctuations of phase.

In the regime $T \gtrsim T_{co}$ the correlation functions can be calculated using the classical field (c-field) approach of [46]. Within this approach the quantum field operators ψ and $\psi^\dagger$ are approximated by the complex (c-number) fields ψ and ψ^* with the grand canonical partition function given by the following path integral:

$$\mathcal{Z} = \int \mathcal{D}\psi\mathcal{D}\psi^* \exp\left(-\frac{1}{k_B T}\int_0^L dx\, \mathcal{H}_c\right), \tag{2.69}$$

where the functional $\mathcal{H}_c(\psi, \psi^*)$ is obtained from $\mathcal{H}\left(\psi, \psi^\dagger\right)$ of Eq. (2.3) by replacing the field operators by the c-valued fields. Notice, that Eq. (2.69) is used instead of the proper field–theoretic partition function

$$\mathcal{Z} = \int \mathcal{D}\psi\mathcal{D}\psi^* \exp\left(\frac{1}{\hbar}\int_0^L\int_0^{\hbar\beta} dx d\tau \left\{i\hbar\psi^*(x,\tau)\frac{\partial\psi}{\partial\tau} - \mathcal{H}_c\left[\psi(x,\tau), \psi^*(x,\tau)\right]\right\}\right), \tag{2.70}$$

in which complex-valued fields ψ, ψ^* also depend on imaginary time τ [73]. In classical field approximation it is assumed that, since temperature is high (β is small), the fields ψ, ψ^* don't vary significantly over the segment $\tau \in [0, \hbar\beta]$ and the integral in τ can be replaced by multiplication by β.

In order to determine the region of applicability of the classical field approximation we notice that eliminating the τ-dependence in the action is equivalent to eliminating all the Matsubara frequency components of the fields apart from the $\omega = 0$ component. For non-interacting bosons that would correspond to replacing the Bose-Einstein by the Rayleigh-Jeans distribution for relevant momenta k,

$$\frac{1}{e^{\beta(E(k)-\mu)} - 1} \to \frac{k_B T}{(E(k) - \mu)}, \tag{2.71}$$

which is valid when occupancy of the low momentum modes is high. For the weakly interacting

Bose gas relevant low momentum modes are Bogoliubov modes with energies smaller than the chemical potential,

$$E(k) \leq g\rho_0. \tag{2.72}$$

By requiring that their occupation is macroscopic we obtain the lower bound on the temperature,

$$g\rho_0 < k_B T. \tag{2.73}$$

On the other hand we must require that the temperature should be lower than the degeneracy temperature,

$$k_B T < k_B T_d \sim \hbar^2 \rho_0^2 / m, \tag{2.74}$$

corresponding to the temperature above which the thermal de-Broglie wavelength becomes smaller than the interparticle distance and the gas becomes non-degenerate at all momenta. This means that the classical field approximation is expected to hold in a broad range of temperatures [46],

$$g\rho_0 < k_B T < \hbar^2 \rho_0^2 / m. \tag{2.75}$$

It is convenient to rescale the field and position variables by setting $\tilde{\psi} = \psi/\psi_0$ and $s = x/x_0$, with

$$\psi_0 = \left(\frac{m k_B^2 T^2}{\hbar^2 g} \right)^{1/6}, \qquad x_0 = \left(\frac{\hbar^4}{m^2 g k_B T} \right)^{1/3}. \tag{2.76}$$

One then finds that the effective action of Eq. (2.69),

$$\frac{1}{k_B T} \int_0^L dx\, \mathcal{H}_c = \int_0^{L/z_0} ds \left(\frac{1}{2} \left| \partial_s \tilde{\psi} \right|^2 + \frac{1}{2} \left| \tilde{\psi} \right|^4 - \eta \left| \tilde{\psi} \right|^2 \right), \tag{2.77}$$

depends on a single dimensionless parameter

$$\eta = \left(\frac{\hbar^2}{mg^2k_B^2T^2}\right)^{1/3}\mu. \tag{2.78}$$

The average density $\rho_0 = \langle\psi^*\psi\rangle$ can be represented as

$$\rho_0 = \left(\frac{mk_B^2T^2}{\hbar^2 g}\right)^{1/3} h(\eta), \tag{2.79}$$

where $h(\eta) = \left\langle\tilde{\psi}^*\tilde{\psi}\right\rangle$ is the "density" of dimensionless fields. Similarly, the phase coherence length Eq. (2.29) can be written as

$$l_\varphi = \frac{\hbar^2\rho_0}{mk_BT} = \left(\frac{\hbar^4}{m^2k_BTg}\right)^{1/3} h(\eta) = x_0 h(\eta), \tag{2.80}$$

which means that given a concrete value of η, both x_0 and l_φ can be used interchangeably to define the scales in the system. This, in turn, means that we can rewrite the one- and two-body correlation functions Eq. (2.4) and Eq. (2.5) as

$$G_1(r_1, r_2) = \rho_0 h_1\left(\frac{x_1}{l_\varphi}, \frac{x_2}{l_\varphi}; \eta\right) \tag{2.81}$$

and

$$G_2(r_1, r_2, r_3, r_4) = \rho_0^2 h_2\left(\frac{x_1}{l_\varphi}, \frac{x_2}{l_\varphi}, \frac{x_3}{l_\varphi}, \frac{x_4}{l_\varphi}; \eta\right), \tag{2.82}$$

where h_1 and h_2 are dimensionless functions.

Similar to Eq. (2.67) we can rewrite the Fourier integral Eq. (2.7) in terms of the dimensionless functions h_1 and h_2 to obtain the regular part of the momentum-momentum correlator

$$\tilde{\mathcal{G}}(k, k') = \frac{l_\varphi}{L}\left(\rho_0 l_\varphi\right)^2 \mathcal{F}\left(2l_\varphi k, 2l_\varphi k'; \eta\right) \tag{2.83}$$

in terms of the dimensionless function $\mathcal{F}(q, q'; \eta)$. This result is similar to Eq. (2.67), only here $\mathcal{F}$ also depends on η and extends the previous result beyond the quasicondensate regime.

In order to find $\mathcal{F}(q, q'; \eta)$ it is still necessary to compute the dimensionless correlators $h(\eta)$, $h_1\left(\frac{x_1}{l_\varphi}, \frac{x_2}{l_\varphi}; \eta\right)$ and $h_2\left(\frac{x_1}{l_\varphi}, \frac{x_2}{l_\varphi}, \frac{x_3}{l_\varphi}, \frac{x_4}{l_\varphi}; \eta\right)$ starting from the action Eq. (2.77). This is the task where the c-field approximation becomes useful [46]. By calling s in Eq. (2.77) "the imaginary time" and by calling the real and imaginary parts of $\tilde{\psi}$,

$$\tilde{\psi} = x + iy, \tag{2.84}$$

"the coordinates", the problem of 1D field theory is mapped onto the problem of 2D quantum mechanics.

Recall that the Feynman path integral [74] allows to express the matrix element of the quantum mechanical evolution operator as

$$\langle x_f | \exp(-iHt/\hbar) | x_i \rangle = \int_{x(0)=x_i}^{x(t)=x_f} \mathcal{D}x(t) \left(\frac{i}{\hbar} \int_0^t dt \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x) \right] \right), \tag{2.85}$$

where

$$U = \exp\left(-\frac{iHt}{\hbar}\right), \qquad H = \frac{p^2}{2m} + V(x) \tag{2.86}$$

are the quantum mechanical evolution and Hamiltonian operators. For the purpose of computing statistical partition function of the same system at temperature $k_B T = 1/\beta$,

$$\mathcal{Z} = \int_{-\infty}^{\infty} dx \, \langle x | \exp(-\beta H) | x \rangle \tag{2.87}$$

one can perform the Wick rotation to the imaginary time in Eq. (2.85),

$$t \to -i\hbar\beta,$$

$$t' \to -i\hbar\tau, \tag{2.88}$$

which gives the following path integral expression for the partition function Eq. (2.87):

$$\mathcal{Z} = \int_{-\infty}^{\infty} dx \int_{x(0)=x}^{x(\hbar\beta)=x} \mathcal{D}x(\tau) \left(- \int_0^{\hbar\beta} d\tau \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right] \right). \tag{2.89}$$

By rewriting the action Eq. (2.77) in terms of x and y of Eq. (2.84) we obtain

$$- \int_0^{L/z_0} ds \left(\frac{1}{2} \left| \partial_s \tilde{\psi} \right|^2 + \frac{1}{2} \left| \tilde{\psi} \right|^4 - \eta \left| \tilde{\psi} \right|^2 \right) =$$

$$= - \int_0^{L/z_0} ds \left[\frac{1}{2} \left(\frac{\partial x}{\partial s} \right)^2 + \frac{1}{2} \left(\frac{\partial y}{\partial s} \right)^2 + \frac{1}{2} \left(x^2 + y^2 \right)^2 - \eta \left(x^2 + y^2 \right) \right] \tag{2.90}$$

and the partition function Eq. (2.69) becomes

$$\mathcal{Z} = \int_{-\infty}^{\infty} dx_0 dy_0 \int \mathcal{D}x(s) \mathcal{D}y(s) \exp \left(- \int_0^{L/z_0} ds \left[\frac{1}{2} \left(\frac{\partial x}{\partial s} \right)^2 + \frac{1}{2} \left(\frac{\partial y}{\partial s} \right)^2 + \frac{1}{2} \left(x^2 + y^2 \right)^2 - \eta \left(x^2 + y^2 \right) \right] \right), \tag{2.91}$$

where the path integral is taken over the paths $x(0) = x\,(L/z_0) = x_0$, $y(0) = y\,(L/z_0) = y_0$ and L takes the role of the inverse temperature. Notice the similarity between Eq. (2.89) and Eq. (2.91). This analogy allows to reduce the problem in field theory to the problem of statistical

physics of the 2D quantum system with the Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}(x^2+y^2)^2 - \eta(x^2+y^2). \tag{2.92}$$

Let us demonstrate how by solving the Schrodinger's equation for the Hamiltonian Eq. (2.92),

$$H\left|\alpha\right\rangle = \epsilon_\alpha \left|\alpha\right\rangle, \tag{2.93}$$

one can find the correlators Eq. (2.81) and Eq. (2.82). Hamiltonian Eq. (2.92) is centrally symmetric and therefore it is convenient to parameterize the eigenstates $|\alpha\rangle$ by the energy and angular momentum quantum numbers n and p,

$$\left\langle r,\theta\left|\alpha\right.\right\rangle = \frac{1}{\sqrt{2\pi}}\phi_n^p(r)e^{ip\theta}, \tag{2.94}$$

where the radial part of the wave function obeys the following eigenvalue equation:

$$\left[-\frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{p^2}{2r^2} + \frac{r^4}{2} - \eta r^2\right]\phi_n^p(r) = \epsilon_n^p\phi_n^p(r), \tag{2.95}$$

and the angular momentum operator is the angular derivative:

$$M\left\langle r,\theta\left|\alpha\right.\right\rangle = -i\frac{\partial}{\partial\theta}\left\langle r,\theta\left|\alpha\right.\right\rangle = p\left\langle r,\theta\left|\alpha\right.\right\rangle. \tag{2.96}$$

The dimensionless correlator $\left\langle\tilde{\psi}^*\left(s_1\right)\tilde{\psi}\left(s_2\right)\right\rangle$ can be written as

$$\left\langle\tilde{\psi}^*\left(s_1\right)\tilde{\psi}\left(s_2\right)\right\rangle = \frac{\mathrm{tr}\left[U_{L-s_1}\tilde{\psi}^*U_{s_1-s_2}\tilde{\psi}U_{s_2}\right]}{tr\left[U_L\right]}, \qquad s_1 \geq s_2, \tag{2.97}$$

$$\left\langle \tilde{\psi}^{*}\left(s_{1}\right)\tilde{\psi}\left(s_{2}\right)\right\rangle =\frac{\mathrm{tr}\left[U_{L-s_{2}}\tilde{\psi}U_{s_{2}-s_{1}}\tilde{\psi}^{*}U_{s_{1}}\right]}{tr\left[U_{L}\right]},\qquad s_{2}>s_{1}, \tag{2.98}$$

where

$$U_{s}=e^{-sH} \tag{2.99}$$

are the imaginary time evolution operators, generated by the Hamiltonian Eq. (2.92). Notice how the order of $\tilde{\psi}^{*}$, $\tilde{\psi}$ and evolution operators is dependent on the ordering of the arguments of the correlator. This is because in the c-field approximation the position "becomes" (imaginary) time and the path integral averaging always computes the normal-ordered average [73].

When written in polar coordinates, the operators

$$\tilde{\psi}=x+iy=re^{i\theta} \tag{2.100}$$

and

$$\tilde{\psi}^{*}=x-iy=re^{-i\theta} \tag{2.101}$$

connect the states with angular momentum quantum number p differing by ±1. This allows to expand the traces in Eqs. (2.97), (2.98) by expanding the evolution operator

$$U_{s}=e^{-sH}=\sum_{\alpha}e^{-s\epsilon_{\alpha}}\left|\alpha\right\rangle \left\langle \alpha\right| \tag{2.102}$$

in the eigenstate basis and inserting the resolution of identity

$$1=\sum_{\alpha}\left|\alpha\right\rangle \left\langle \alpha\right| \tag{2.103}$$

where necessary (the states $|\alpha\rangle$ are always assumed to be normalized). By taking the thermodynamic limit, $L\rightarrow\infty$, the summation in expressions Eq. (2.102) for U_{L}, $U_{L-s_{1}}$ and $U_{L-s_{2}}$ can

be restricted to the ground state of Eq. (2.92) with $p = n = 0$, which we also denote $|0\rangle$,

$$H|0\rangle = \epsilon_0 |0\rangle , \tag{2.104}$$

so that

$$U_L \approx e^{-L\epsilon_0} |0\rangle \langle 0| , \qquad U_{L-s_1} \approx e^{-(L-s_1)\epsilon_0} |0\rangle \langle 0| , \tag{2.105}$$

since the terms corresponding to the excited states are exponentially small. The thermodynamic limit approximation Eq. (2.105) restricts the traces Eqs. (2.97), (2.98) to the ground state averages

$$\left\langle \tilde{\psi}^* (s_1) \tilde{\psi} (s_2) \right\rangle = \frac{e^{-(L-s_1)\epsilon_0} e^{-s_2\epsilon_0} \sum_\alpha e^{-(s_1-s_2)\epsilon_\alpha} \langle 0| re^{-i\theta} |\alpha\rangle \langle \alpha| re^{i\theta} |0\rangle}{e^{-L\epsilon_0}} =$$

$$= e^{(s_1-s_2)\epsilon_0} \sum_\alpha e^{-(s_1-s_2)\epsilon_\alpha} |A_{\alpha 0}|^2 , \qquad s_1 \geq s_2,$$

$$\left\langle \tilde{\psi}^* (s_1) \tilde{\psi} (s_2) \right\rangle = e^{(s_2-s_1)\epsilon_0} \sum_\alpha e^{-(s_2-s_1)\epsilon_\alpha} |A_{\alpha 0}|^2 , \qquad s_2 \geq s_1, \tag{2.106}$$

or, summarizing both cases,

$$\left\langle \tilde{\psi}^* (s_1) \tilde{\psi} (s_2) \right\rangle = \sum_\alpha e^{-i|s_1-s_2|(\epsilon_\alpha-\epsilon_0)} |A_{\alpha 0}|^2 . \tag{2.107}$$

Here

$$A_{\alpha\beta} = \langle \alpha| re^{i\theta} |\beta\rangle \propto \delta_{p_\alpha, p_\beta+1} \tag{2.108}$$

is the matrix element of the operator $\tilde{\psi}$, Eq. (2.100). As was already mentioned, due to presence of factor $e^{i\theta}$, the operator $\tilde{\psi}$ can only connect the ground state ($p = 0$) with excited states with

$p = 1$, which restricts the summation in Eq. (2.107),

$$\left\langle \tilde{\psi}^* (s_1) \tilde{\psi} (s_2) \right\rangle = \sum_n e^{-|s_1 - s_2|(\epsilon_n^1 - \epsilon_0)} \left| \left\langle \phi_n^1 \right| r \left| 0 \right\rangle \right|^2 . \tag{2.109}$$

In Eq. (2.109) the states with higher n are not important, since their contribution goes exponentially small with their energy. Summation in Eq. (2.109) is further reduced by the fact, that states with different n-s have very different nodal structure and the matrix element $\langle \varphi_n^1 | \, r \, | 0 \rangle$ becomes negligible for high n. In fact, we keep only a single term in the sum in Eq. (2.109) corresponding to $n = 0$, $p = 1$:

$$G_1(s_1, s_2) = \left\langle \tilde{\psi}^* (s_1) \tilde{\psi} (s_2) \right\rangle = e^{-|s_1 - s_2|(\epsilon_0^1 - \epsilon_0)} \left| \left\langle \phi_0^1 \right| r \left| 0 \right\rangle \right|^2 . \tag{2.110}$$

The two–particle correlator

$$G_2(s_1, s_2, s_3, s_4) = \left\langle \tilde{\psi}^* (s_1) \tilde{\psi} (s_2) \tilde{\psi}^* (s_3) \tilde{\psi} (s_4) \right\rangle \tag{2.111}$$

can be calculated in a similar manner, starting with four–operator analogues of Eqs. (2.97), (2.98), and writing down an expression for each possible ordering of s_1, s_2, s_3 and s_4. Having more field operators adds complexity, there are now 24 different orderings of s_i. However by making use of symmetries of G_2, namely its invariance under exchanges

$$s_1 \rightleftarrows s_3 \tag{2.112}$$

and

$$s_2 \rightleftarrows s_4, \tag{2.113}$$

and under simultaneous exchange

$$s_1 \rightleftarrows s_2, \qquad s_3 \rightleftarrows s_4 \tag{2.114}$$

we can reduce the problem to computing just 3 distinct orderings,

$$s_1 > s_2 > s_3 > s_4, \tag{2.115}$$

$$s_1 > s_2 > s_4 > s_3, \tag{2.116}$$

and

$$s_1 > s_3 > s_2 > s_4, \tag{2.117}$$

and evaluate the remaining 21 by applying the symmetries, Eqs. (2.112), (2.113) and (2.114).

For example, $G_2(s_1, s_2, s_3, s_4)$ corresponding to the ordering Eq. (2.115) is given by

$$\frac{\mathrm{tr}\left[U_{L-s_1}\tilde{\psi}^* U_{s_1-s_2}\tilde{\psi} U_{s_2-s_3}\tilde{\psi}^* U_{s_3-s_4}\tilde{\psi} U_{s_4}\right]}{tr\left[U_L\right]}. \tag{2.118}$$

Expanding the evolution operators Eq. (2.102), inserting the resolutions of identity Eq. (2.103) and taking the thermodynamic limit Eq. (2.105), the expression Eq. (2.118) becomes

$$\sum_{\alpha\beta\gamma} e^{-\epsilon_0(s_4\text{-}s_1)-\epsilon_\gamma(s_3\text{-}s_4)-\epsilon_\beta(s_2\text{-}s_3)-\epsilon_\alpha(s_1\text{-}s_2)} A^*_{\alpha 0} A_{\alpha\beta} A^*_{\gamma\beta} A_{\gamma 0}, \tag{2.119}$$

where again for non–zero product of matrix elements the states γ and α must have angular momentum quantum number $p = 1$ and state β must have $p = 0$. Just like in Eq. 2.109, the states with higher n-s are suppressed due to the exponential prefactor, so that summation may

be restricted to a single state for each index and Eq. 2.119 becomes

$$G_2(s_1,s_2,s_3,s_4) = e^{-(\epsilon_0^1-\epsilon_0)(s_1-s_2+s_3-s_4)} \left|\left\langle \phi_0^1 \right| r \left|0\right\rangle\right|^4, \qquad s_1 > s_2 > s_3 > s_4. \tag{2.120}$$

Similar calculations can be done for other orderings, resulting in

$$G_2(s_1,s_2,s_3,s_4) = e^{-(\epsilon_0^1-\epsilon_0)(s_1-s_2-s_3+s_4)} \left|\left\langle \phi_0^1 \right| r \left|0\right\rangle\right|^4, \qquad s_1 > s_2 > s_4 > s_3 \tag{2.121}$$

and

$$G_2(s_1,s_2,s_3,s_4) = e^{-(\epsilon_0^1-\epsilon_0)(s_1-s_4)-(\epsilon_0^2-\epsilon_0^1)(s_3-s_2)} \left|\left\langle \phi_0^1 \right| r \left|0\right\rangle\right|^2 \left|\left\langle \phi_0^2 \right| r \left|\phi_0^1\right\rangle\right|^2, \qquad s_1 > s_3 > s_2 > s_4. \tag{2.122}$$

Finally, results for G_2 and G_1 are plugged in Eq. 2.7, the Fourier transforms are taken and the singular part is subtracted, resulting in the regular part

$$\mathcal{F}\left(2l_\varphi k, 2l_\varphi k'; \eta\right)$$

as a function of η. The main result of this section, $\mathcal{F}\left(2l_\varphi k, 2l_\varphi k'; \eta\right)$, demonstrates how correlations change in the region of crossover from the ideal Bose gas ($\eta \ll -1$) to the quasicondensate ($\eta \gg 1$), see Figs. 2.8 and 2.9.

It is interesting to study how the ideal Bose gas and quasicondensate limits, $\eta \ll -1$ and $\eta \gg 1$, are recovered, and how results of Section 2.3 are approached. The qualitative picture is that of the double well $\longleftrightarrow$ single well potential, depending on the sign and magnitude of η see Fig. 2.10,

$$V(r) = \frac{r^4}{2} - \eta r^2, \tag{2.123}$$

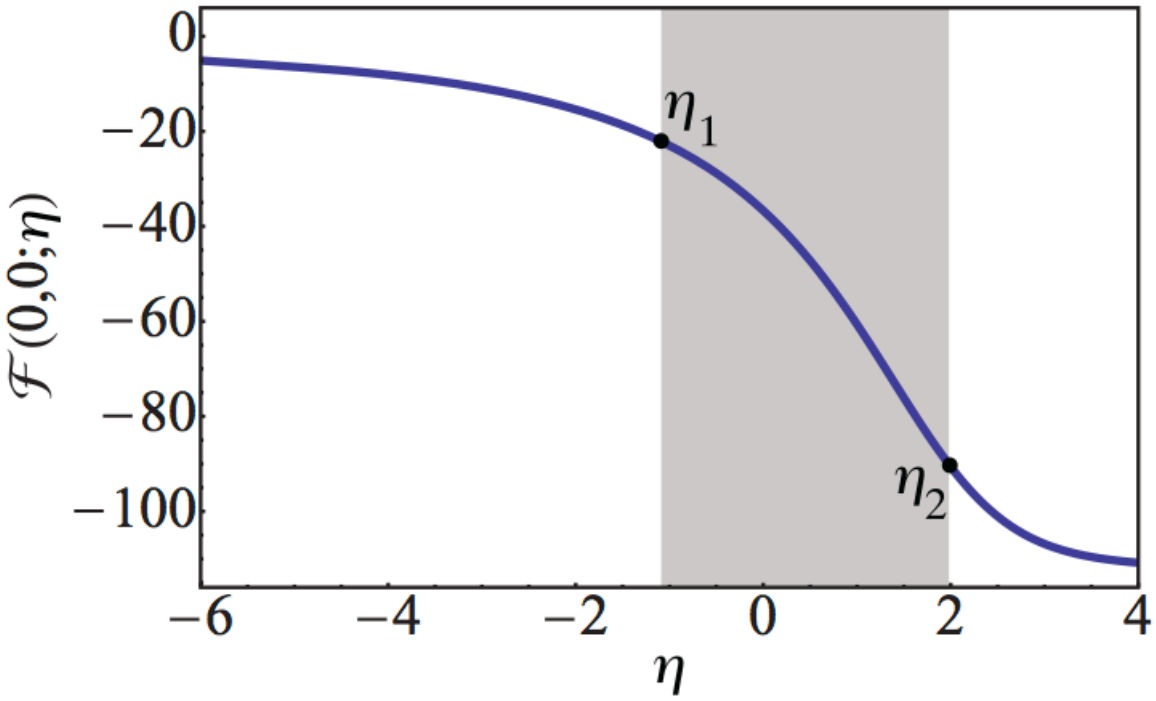


Figure 2.8: The minimum of $\mathcal{F}(q,-q,\eta)$ as the parameter η is varied from $\eta=-\infty$, corresponding to highly degenerate ideal Bose gas, to $\eta=\infty$, corresponding to the phase-fluctuating quasicondensate. The shaded area represents the crossover region between those two extremes, such that at η_1 and η_2 $\mathcal{F}(0,0,\eta)$ is, respectively, 20% and 80% of its value in the quasicondensate.

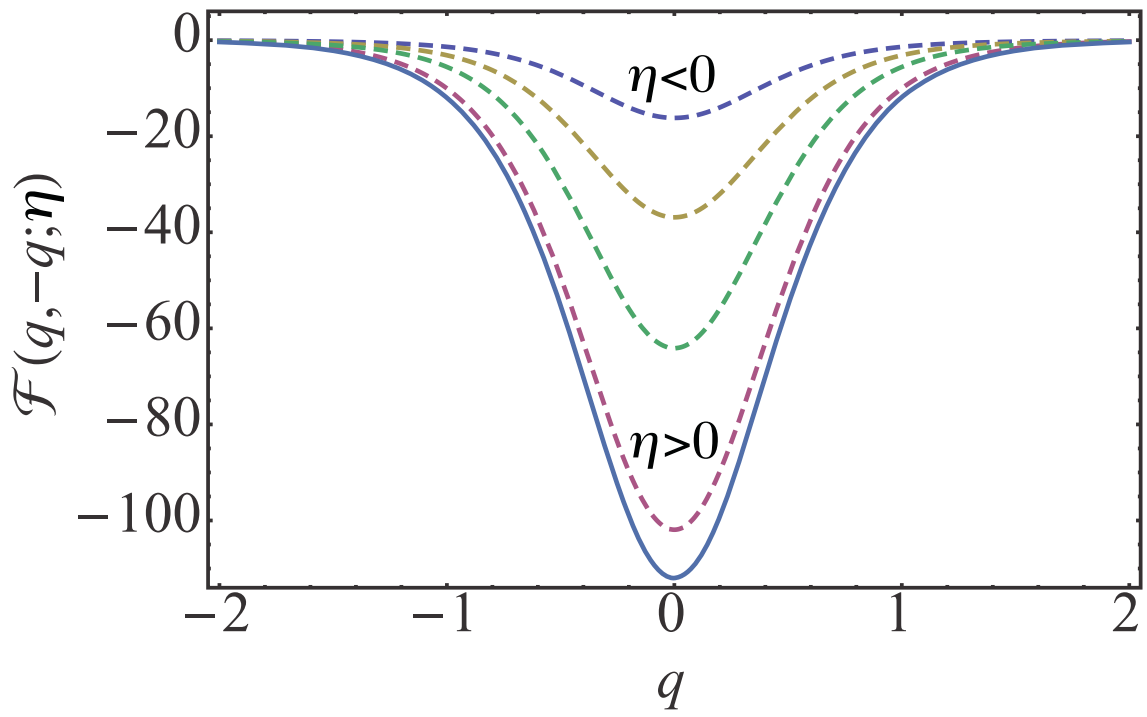


Figure 2.9: Correlation function $\mathcal{F}(q, q', \eta)$ evaluated at the opposite momenta values $q' = -q$, $\mathcal{F}(q, -q, \eta)$, plotted for different η. $\eta \gg 0$ corresponds to the quasicondensate regime, while $\eta \ll 0$ corresponds to highly degenerate Bose gas.

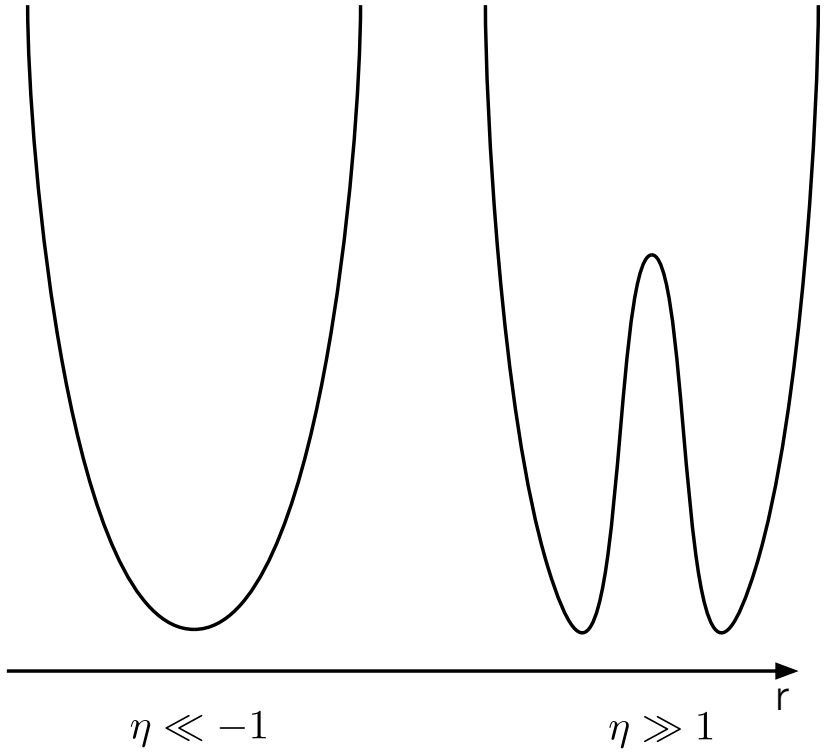


Figure 2.10: Schematic plot of the potential energy of the effective quantum mechanical problem, Eq. (2.10), symmetrically continued to the negative values of r for convenience. For large positive values of η (positive chemical potential) the system sits at $r \approx r_0 = \sqrt{\eta}$, this corresponds to quasicondensate regime. For large negative η (corresponds to negative chemical potential) the system spends most of its time near the centre experiencing almost quadratic potential, this corresponds to the degenerate Bose gas.

compare with Eq. (2.92) and notice that the centrifugal term $p^2/2r^2$ is omitted. It is not important for the discussion that follows as wave functions with non-zero p vanish at the origin,

$$\phi_n^{p\neq 0}(r) \to 0, \qquad r \to 0. \tag{2.124}$$

First, consider the limit $\eta \gg 1$. This corresponds to the double-well potential Eq. (2.123) and wave functions Eq. (2.93) corresponding to the lowest–lying states significantly differ from zero only in the region $r \approx r_0 = \sqrt{\eta}$. This has the following consequences on the c-field

calculations. First is that in the 2D Hamiltonian Eq. (2.92) the azimuthal kinetic energy

$$H_\theta = -\frac{1}{2r^2}\frac{\partial^2}{\partial\theta^2} \approx -\frac{1}{2r_0^2}\frac{\partial^2}{\partial\theta^2} = \frac{p^2}{2\eta} \tag{2.125}$$

approximately becomes r-independent and the Hamiltonian Eq. (2.92) separates into the azimuthal and radial parts. Accordingly, for the matrix element Eq. (2.108) we obtain

$$A_{\alpha\alpha'} = \delta_{p_\alpha, p_{\alpha'}+1} \left\langle \phi_n^p \right| r \left| \phi_{n'}^{p'} \right\rangle \approx r_0 \delta_{p_\alpha, p_{\alpha'}+1} \left\langle \phi_n^p \right| \left. \phi_{n'}^{p'} \right\rangle = \sqrt{\eta}\delta_{p_\alpha, p_{\alpha'}+1}\delta_{nn'} \tag{2.126}$$

which allows to restrict summations in Eqs. (2.109) and (2.119) to a single terms with $n_\alpha = ... = 0$ only.

Also, the fact that the energy eigenvalues separate into p-independent and angular parts,

$$\epsilon_n^p = \epsilon_n + \frac{p^2}{2\eta}, \tag{2.127}$$

allows us to calculate the exponents in Eq. (2.109) analytically. The only relevant energy differences are

$$\epsilon_0^1 - \epsilon_0^0 = \frac{1}{2\eta} \tag{2.128}$$

and

$$\epsilon_0^2 - \epsilon_0^1 = \frac{3}{2\eta}. \tag{2.129}$$

Using these results, we find that $G_1(s_1, s_2)$ Eq. (2.109) reduces to

$$\left\langle \tilde{\psi}^* (s_1) \tilde{\psi} (s_2) \right\rangle = \eta e^{-|s_1 - s_2|/2\eta}. \tag{2.130}$$

Going back to the natural units, using Eqs. (2.76), (2.78) and (2.79), we obtain the quasicon-

densate equation of state

$$\rho_0 = \frac{\mu}{g} \tag{2.131}$$

and one–particle correlator

$$G_1(x_1, x_2) = \frac{\mu}{g} \exp\left(-\frac{|x_1 - x_2|}{2\eta} \left(\frac{m^2 g k_B T}{\hbar^4} \right)^{1/3} \right) =$$

$$= \frac{\mu}{g} \exp\left(-\frac{|x_1 - x_2|}{2} \left(\frac{m g^2 k_B^2 T^2}{\hbar^2} \right)^{1/3} \left(\frac{m^2 g k_B T}{\hbar^4} \right)^{1/3} \frac{1}{\mu} \right). \tag{2.132}$$

The terms in the exponent multiply out to give

$$\left(\frac{m g^2 k_B^2 T^2}{\hbar^2} \right)^{1/3} \left(\frac{m^2 g k_B T}{\hbar^4} \right)^{1/3} \frac{1}{\mu} = \frac{m g k_B T}{\hbar^2 \mu} = \frac{m k_B T}{\hbar^2 \rho_0} = \frac{1}{l_\varphi}, \tag{2.133}$$

the inverse quasicondensate coherence length of Eq. (2.29). Thus Eq. (2.132) recovers the result Eq. (2.62) for the quasicondensate regime. Similar, albeit more lengthy, calculations performed for $G_2(s_1, s_2, s_3, s_4)$ recover the results of Section 2.3 and we can make the identification

$$\mathcal{F}(q, q'; +\infty) = \mathcal{F}(q, q') . \tag{2.134}$$

In the opposite limit, $\eta \ll -1$, the potential Eq. (2.123) is concentrated near the origin $r = 0$ and so are the ground and lower excited states. In fact, for such low-lying states the potential in Eqs. (2.123), (2.92) is dominated by the quadratic term,

$$H \approx \frac{1}{2} \left(p_x^2 + p_y^2 \right) + |\eta| \left(x^2 + y^2 \right), \tag{2.135}$$

i.e. the problem becomes that of a harmonic oscillator with frequency $\omega = \sqrt{2|\eta|}$ in $2D$. Then all the matrix elements and energy eigenvalues can be obtained by standard means [75]. In

particular, for the first excited state $|\alpha\rangle$ with $n_\alpha = 0$ and $p_\alpha = 1$ we have

$$\langle\alpha|\, x\, |0\rangle = \langle\alpha|\, y\, |0\rangle = (2\omega)^{-1/2} \tag{2.136}$$

and

$$\epsilon_\alpha - \epsilon_0 = \omega = \sqrt{2\,|\eta|}, \tag{2.137}$$

so that one-particle correlator Eq. (2.109) becomes

$$G_1(s_1, s_2) = e^{-(\epsilon_\alpha - \epsilon_0)|s_1 - s_2|}\, |\langle\alpha|\, x + iy\, |0\rangle|^2 = \frac{1}{\sqrt{2\eta}} e^{-\sqrt{2|\eta|}|s_1 - s_2|}. \tag{2.138}$$

Direct calculation of G_2 is, again, more elaborate because of many orderings one has to consider, see Eqs. (2.120)-(2.122). In the end, however, the result consistent with Wick's theorem is recovered,

$$G_2(s_1, s_2, s_3, s_4) = G_1(s_1, s_2)G_1(s_3, s_4) + G_1(s_1, s_4)G_1(s_3, s_2), \tag{2.139}$$

an expected outcome for the quadratic theory. Obviously, the same relation holds for dimensional fields, and Eq. (2.139) can be directly compared to Eq. (2.21). Thus we recover the result that in the ideal Bose gas, the two-body momentum correlator is given simply by its singular part and the regular part $\tilde{G}_2$ vanishes identically. Notice, that we have neglected the term containing the δ-function in Eq. (2.21),

$$G_1(x_1 - x_4)\delta(x_2 - x_3), \tag{2.140}$$

since it can be neglected in the highly degenerate ideal Bose gas regime, considered here.

2.5 Experimental considerations

To measure the correlator $\mathcal{G}(k, k')$ discussed in this chapter, one needs to have access to a set of momentum distributions of a 1D system. This can be done by first releasing the transverse trap that is used to restrict the system to 1D in the first place, thus eliminating the inter–atomic interactions. This will not change the momentum distribution in the longitudinal direction since the typical time it takes to switch off the transverse trap is much smaller than the relevant longitudinal timescale[2]. After that one can e.g. perform the time-of-flight measurement of the momentum distribution by optical imaging of the expanding cloud.

In ultracold atom experiments it is usually not possible to resolve the states with momenta separated by $\Delta k = 2\pi/L$, so rather than accessing n_k and its distribution functions directly, one gets access to N_k - integrated number of particles in a detection "bin" centered at momentum k, and its correlations functions $\langle N_k \rangle$, $\langle N_k N_{k'} \rangle$, etc. If we assume that the size of detection bin Δ_k satisfies

$$\frac{1}{L} \ll \Delta_k \ll \frac{1}{l_\varphi}, \tag{2.141}$$

so that it is both too large to resolve individual momentum states and small enough not to extend across the bulk of the momentum distribution, the average number of atoms in a detection bin is

$$\langle N_k \rangle = \frac{L\Delta_k}{2\pi} \langle n_k \rangle , \tag{2.142}$$

the number of atoms in a momentum state with k belonging to the bin, times the number of such states per bin, $\Delta_k/(2\pi/L)$.

[2]The ratio of trap switching time to the characteristic longitudinal time is equal to the ratio of frequencies of longitudinal and transverse traps $\omega_{long}/\omega_{trans} \ll 1$

The two-bin correlation function is given by

$$\langle N_k N_{k'}\rangle - \langle N_k\rangle \langle N_{k'}\rangle = \langle N_k\rangle \delta_{kk'} + \langle N_k\rangle \langle N_{k'}\rangle \left[\frac{2\pi}{L\Delta_k}\delta_{kk'} + \frac{l_\varphi}{L} f(2l_\varphi k, 2l_\varphi k')\right] \tag{2.143}$$

where the first term is the shot noise, the second is the bunching term, compare to Eq. (2.22), and the third term corresponds to the regular part $\tilde{\mathcal{G}}(k,k')$ with $f(q,q')$ being the normalized version of the universal function $\mathcal{F}$,

$$\tilde{\mathcal{G}}(k,k') = \frac{l_\varphi}{L}\left(\rho_0 l_\varphi\right)^2 \mathcal{F}\left(2l_\varphi k, 2l_\varphi k'\right)$$

$$\downarrow$$

$$\frac{\tilde{\mathcal{G}}(k,k')}{\langle n_k\rangle \langle n_{k'}\rangle} = \frac{l_\varphi}{L} f\left(2l_\varphi k, 2l_\varphi k'\right), \tag{2.144}$$

$$f(q,q') = \frac{\mathcal{F}(q,q')(1+q^2)(1+q'^2)}{16}. \tag{2.145}$$

The shot noise term is negligible compared to the bunching term when $\langle n_k\rangle \gg 1$, which is the case for all k-s belonging to the bulk of the momentum distribution, $k \leq 1/l_\varphi$, so we may safely neglect it under usual the experimental condition Eq. (2.141).

Comparing the bunching term and the term corresponding to the regular part of the correlator at $k = k'$, we find that their ratio does not depend on L and therefore is finite in the thermodynamic limit. On the other hand, the ratio of the second to the third term is of the order $(\Delta_k l_\varphi)^{-1}$, which according to Eq. (2.141) is much greater than 1, so good experimental precision is required. It was demonstrated in a number of recent high–precision measurements in cold atom that weak signals like this, of the shot-noise smallness or even weaker, can be measured, see e.g. [76].

Only the regular part of Eq. (2.143) contributes to the cross-correlation of different bins,

$k \neq k'$, and it scales as $1/L$. Again, it should be possible to measure such weak signal with the current state-of-the-art techniques, as demonstrated in [76].

Finally, the region of applicability of the classical field approximation was discussed in Section 2.4. It is expected to hold for temperatures above the chemical potential but below the degeneracy temperature,

$$g\rho_0 < k_B T < \hbar^2 \rho_0^2 / m. \tag{2.146}$$

2.6 Summary

We have calculated the two-body momentum correlation function for a 1D weakly interacting Bose gas. After demonstrating what to expect in the case of coherent condensate in Section 2.2, the result for the phase fluctuating quasicondensate was derived analytically using the Luttinger Liquid theory and the regular part of the correlator was expressed in terms of the universal dimensionless function $\mathcal{F}(q, q')$, Eq. (2.68). As it turned out, in the quasicondensate regime the momentum-momentum correlator is non–zero everywhere, unlike demonstrated by the Bogoliubov analysis for the phase-coherent condensate, where the only correlations present are between same and opposite momenta. Also, zones of negative correlations have emerged, see Figs. 2.5 and 2.6.

We have then used the classical field approximation to the field theory to numerically investigate the behavior of $\tilde{\mathcal{G}}(k, k')$ in the regime intermediate between the quasicondensate and the degenerate Bose gas. This has allowed us to trace how $\tilde{\mathcal{G}}(k, k')$ varies smoothly from

$$\tilde{\mathcal{G}}(k, k') = 0 \tag{2.147}$$

in the strongly degenerate Bose gas to it's quasicondensate value of

$$\tilde{\mathcal{G}}(k,k') \propto \mathcal{F}(2l_\varphi k, 2l_\varphi k'). \tag{2.148}$$

We note that calculating the momentum correlations in the strongly interacting regime would require different approach and remains an open problem.

Chapter 3

Density distribution in the 1D Bose gas

In this Chapter we apply the numerical calculations done for the Section 2.4 of Chapter 2 (Classical field approximation) to the problem of particle number statistics.

3.1 Introduction to the problem: a toy model

Consider a classical uniform noninteracting 1D gas consisting of N particles, confined to a box of size L and held at constant temperature T, i.e. a classical canonical ensemble. One may ask the following question: if one specifies a certain segment of length $l \leq L$, what are the probabilities w_p of observing 0, 1, 2, ..., p, ... particles in a pixel of that size? The answer follows from basic probabilistic arguments [77]. Homogeneity implies that for a specific particle the probability of being observed on that segment is l/L and lack of interactions implies that probabilities for each particle are independent. Therefore, the probability w_p is given by the binomial distribution

$$w_p = \frac{N!}{p!\,(N-p)!} \left(\frac{l}{L}\right)^p \left(1 - \frac{l}{L}\right)^{N-p} \tag{3.1}$$

with the average

$$\langle p \rangle = \frac{l}{L} N = l \langle \rho \rangle \tag{3.2}$$

and the variance

$$\left\langle \delta p^2 \right\rangle = N \frac{l}{L} \left(1 - \frac{l}{L} \right) = \langle p \rangle - \langle p \rangle \frac{l}{L}. \tag{3.3}$$

In thermodynamic limit

$$N \to \infty,\, L \to \infty,\, \langle p \rangle = const \tag{3.4}$$

the binomial distribution Eq. (3.1) converges to the Poisson distribution

$$w_p \to \frac{\langle p \rangle^p \, e^{-\langle p \rangle}}{p!}. \tag{3.5}$$

It turns out that w_p is Poissonian even if one relaxes the condition $N = const$ and instead treats the system as grand canonical ensemble with $\mu = const$ [77]. Then the distribution is given by

$$w_p = \frac{\left(l e^{\beta\mu} / \lambda \right)^p \, e^{-l e^{\beta\mu}/\lambda}}{p!} \tag{3.6}$$

with the average

$$\langle p \rangle = l e^{\beta\mu} / \lambda \tag{3.7}$$

and the variance

$$\left\langle \delta p^2 \right\rangle = l e^{\beta\mu} / \lambda, \tag{3.8}$$

with λ being the de-Broglie wavelength at the given temperature.

3.2 Density distribution in real systems

Measurements of particle density fluctuations in 1D ultracold atomic gases have been recently described in the literature [61, 76]. Experimental approach used was to divide the trapped 1D gas into pixels and measure how particle density fluctuates in each of them using absorption imaging. Moments of density distribution function can give access to key quantities that characterize ultracold systems, for example the third moment carries information about the strength of three-body correlations, so extending the toy calculations of Section 3.1 to real systems is a useful undertaking, but not nearly as trivial since quantum mechanics and inter particle interactions become important.

Certain quantities related to the particle number distribution can be obtained theoretically using the Fluctuation-Dissipation Theorem. For example, if the length l of the pixel is large enough so that it can be treated as a thermodynamic system at temperature $k_B T = 1/\beta$ and chemical potential μ, then moments of the particle number distribution can be found from the grand potential Φ,

$$\Phi = -\frac{1}{\beta} \ln \Xi, \tag{3.9}$$

or equivalently, the grand partition function Ξ of the pixel. The average number of particles is

$$\langle p \rangle = -\left(\frac{\partial \Phi}{\partial \mu}\right)_{T,l} = \frac{1}{\beta \Xi} \left(\frac{\partial \Xi}{\partial \mu}\right)_{T,l}, \tag{3.10}$$

and similarly for higher moments of p:

$$\left\langle p^2 \right\rangle = \frac{1}{\beta^2 \Xi} \left(\frac{\partial^2 \Xi}{\partial \mu^2}\right)_{T,l}, \tag{3.11}$$

$$\left\langle p^3 \right\rangle = \frac{1}{\beta^3 \Xi} \left(\frac{\partial^3 \Xi}{\partial \mu^3}\right)_{T,l}. \tag{3.12}$$

The variance of p, from Eqs. (3.10) and (3.11), is

$$\left\langle \delta p^2 \right\rangle = \frac{1}{\beta} \frac{\partial \left\langle p \right\rangle}{\partial \mu}. \tag{3.13}$$

Thus knowledge of the equation of state,

$$\rho_0 = \rho_0(T, \mu), \qquad \langle p \rangle = l\rho_0(T, \mu) \tag{3.14}$$

allows one to obtain the higher moments of p. For quantum gases the equation of state can be obtained e.g. using Yang-Yang thermodynamics [33]. As was mentioned before, the only condition is for the pixel to have sufficient size for thermodynamics to be applicable, which for quantum gases means $l > l_\varphi$, the size must be larger than the coherence length [76].

3.3 Classical Field approach to calculating the density distribution

In Chapter 2 we considered a system of 1D bosons with repulsive contact interactions, with Hamiltonian Eq. (2.3), and, specifically, in Section 2.4 we showed how various field correlators can be calculated approximately. The averages corresponding to the particle number on a segment

$$\langle p \rangle = \int_0^l dx \, \left\langle \psi^\dagger(x)\psi(x) \right\rangle \tag{3.15}$$

and to higher moments $\left\langle p^k \right\rangle$ are also expressed in terms of such correlators and we are going to apply the classical field approximation of Section (2.4) to compute them.

Recall that the idea behind the classical field approximation is to disregard the integration

in imaginary time τ in the action Eq. (2.70), replacing it by multiplication

$$\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau\, F\left[\psi^*(\tau),\psi(\tau)\right] \to \beta F\left[\psi^*,\psi\right], \tag{3.16}$$

and then to reinterpret the coordinate x as the imaginary time and real and imaginary parts of the fields as the coordinates (x,y) in the 2D space, compare to Eq. (2.84):

$$\tilde{\psi}(s) = x + iy, \qquad \tilde{\psi}(s)^* = x - iy, \tag{3.17}$$

thus mapping the problem in 1D statistical physics onto quantum mechanics of a point mass in 2D. For systems of large size L the imaginary "time" runs from zero to almost "infinity", i.e. the effective $\beta\to\infty$. The averages then become almost ground state averages. In Eq. (3.17) we have also used the rescaled, dimensionless, fields and the distances (see Eq. (2.76)),

$$\psi = \left(\frac{mk_B^2T^2}{\hbar^2 g}\right)^{1/6}\tilde{\psi}. \tag{3.18}$$

Then the classical field partition function

$$\mathcal{Z} = \int \mathcal{D}\tilde{\psi}\mathcal{D}\tilde{\psi}^* \exp\left(-\int_0^{L/z_0} ds\, \left(\frac{1}{2}\left|\partial_s\tilde{\psi}\right|^2 + \frac{1}{2}\left|\tilde{\psi}\right|^4 - \eta\left|\tilde{\psi}\right|^2\right)\right), \tag{3.19}$$

or equivalently, in the x, y notation,

$$\mathcal{Z} = \int \mathcal{D}x\mathcal{D}y \exp\left(-\int_0^{L/z_0} ds\, \left[\frac{1}{2}\left(\frac{\partial x}{\partial s}\right)^2 + \frac{1}{2}\left(\frac{\partial y}{\partial s}\right)^2 + \frac{1}{2}\left(x^2+y^2\right)^2 - \eta\left(x^2+y^2\right)\right]\right), \tag{3.20}$$

depends on a single parameter

$$\eta = \left(\frac{\hbar^2}{mg^2k_B^2T^2}\right)^{1/3}\mu. \tag{3.21}$$

We start by considering the following quantity,

$$\left\langle \delta\left(\rho - \psi^\dagger\psi\right)\right\rangle. \tag{3.22}$$

Eq. (3.22) is the operator average performed over the grand canonical ensemble. The combination $\psi^\dagger\psi$ stands for the particle density operator and the meaning of average Eq. (3.22) is that of density distribution in the system. Indeed, by formally considering the expression

$$w(\rho) = \left\langle \delta\left(\rho - \psi^\dagger\psi\right)\right\rangle, \tag{3.23}$$

using the properties of δ-function we find,

$$\int w(\rho)\, d\rho = \left\langle \int \delta(\rho - \psi^\dagger\psi)\, d\rho \right\rangle = 1, \tag{3.24}$$

so that $w(\rho)$ is correctly normalized to 1, as any distribution should be. In a similar fashion,

$$\int \rho w(\rho)\, d\rho = \left\langle \int \rho\delta(\rho - \psi^\dagger\psi)\, d\rho \right\rangle = \left\langle \psi^\dagger\psi \right\rangle, \tag{3.25}$$

$$\int \rho^2 w(\rho)\, d\rho = \left\langle \int \rho^2\delta(\rho - \psi^\dagger\psi)\, d\rho \right\rangle = \left\langle \psi^\dagger\psi\psi^\dagger\psi \right\rangle = \left\langle \left(\psi^\dagger\psi\right)^2 \right\rangle, \tag{3.26}$$

$$\dots$$

so that $w(\rho)$ behaves as a distribution function indeed.

We apply the classical field approximation to Eq. (3.23) by going to the path integral

notation instead of operators,

$$w(\tilde{\rho}) = \left\langle \delta\left(\tilde{\rho} - \tilde{\psi}^*\tilde{\psi}\right)\right\rangle, \tag{3.27}$$

where without loss of generality we use rescaled fields $\tilde{\psi}$. Using the 2D quantum mechanics correspondence Eq. (3.27) becomes

$$w(\tilde{\rho}) = \left\langle \delta\left(\tilde{\rho} - x^2 - y^2\right)\right\rangle, \tag{3.28}$$

where the average is evaluated in the ground state of the $2D$ quantum system described by the following Schrodinger's equation:

$$-\frac{1}{2}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) + \left(\frac{\left(x^2+y^2\right)^2}{2} - \eta\left(x^2+y^2\right)\right)\varphi = E\varphi. \tag{3.29}$$

The probability of $r^2 = x^2 + y^2$ being equal to $\tilde{\rho}$ in the ground state of effective quantum mechanicanical problem is just the square of the normalized ground state wave function

$$w(\tilde{\rho}) = \pi\left|\phi_0^0(\sqrt{\tilde{\rho}})\right|^2. \tag{3.30}$$

Ground state wave functions have been computed for a wide range of values of η, as part of Section 2.4. It would seem that we have a result, but some discussion is still in place.

3.4 Making sense of $w(\rho)$

Notice, that we said "formally" before Eq. (3.24) for a reason, and a word of caution is in place. Consider a uniform gas of noninteracting bosons, then the moments of density can be evaluated

directly with the aid of bosonic commutation relation,

$$\left[\psi(x), \psi^\dagger(x')\right] = \delta(x - x') \tag{3.31}$$

and Wick's theorem,

$$\begin{aligned}
\left\langle \left(\psi^\dagger\psi\right)^2 \right\rangle &= \lim_{x'\to x} \left\langle \psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x') \right\rangle \\
&= \lim_{x'\to x} \left(\left\langle \psi^\dagger(x)\psi^\dagger(x')\psi(x)\psi(x') \right\rangle + \left\langle \psi^\dagger(x)\psi(x') \right\rangle \delta(x - x') \right) \\
&= 2\left\langle \psi^\dagger\psi \right\rangle^2 + \left\langle \psi^\dagger\psi \right\rangle \delta(0),
\end{aligned} \tag{3.32}$$

and similar for the higher moments. Notice the $\delta(0)$ term in Eq. (3.32) - it shows that the distribution of the actual particle density is singular, and when writing expressions like Eq. (3.23) we should always keep some length scale in mind, i.e. we should apply Eq. (3.23) not to $\psi^\dagger\psi$, but rather to $\int_0^l \psi^\dagger(x)\psi(x)\,dx$,

$$w(p) = \left\langle \delta\left(p - \int_0^l \psi^\dagger(x)\psi(x)\,dx \right) \right\rangle, \tag{3.33}$$

so that we are calculating statistics of the number of particles, which is well defined, and which is the quantity that gets measured experimentally as was mentioned in Section 3.2.

That quantity Eq. (3.33) is well defined can be demonstrated by repeating the calculations analogous to Eq. (3.32) for the particle number

$$p = \int_0^l \psi^\dagger(x)\psi(x)\,dx, \tag{3.34}$$

rather than density.

$$\begin{aligned}\left\langle p^2\right\rangle &= \left\langle \int_0^l \int_0^l \psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x')\,dxdx' \right\rangle = \int_0^l \int_0^l \left\langle \psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x')\right\rangle dxdx' \\ &= \int_0^l \int_0^l \left(\left\langle \psi^\dagger(x)\psi(x')\right\rangle \left\langle \psi^\dagger(x')\psi(x)\right\rangle + \left\langle \psi^\dagger(x)\psi(x)\right\rangle \left\langle \psi^\dagger(x')\psi(x')\right\rangle + \right. \\ &\quad \left. + \left\langle \psi^\dagger(x)\psi(x')\right\rangle \delta(x-x')\right) dxdx'. \end{aligned} \tag{3.35}$$

The first two terms under the integral on the right-hand side are regular, as before; for non-interacting bosons, assuming large negative chemical potential μ,

$$\left\langle \psi^\dagger(x)\psi(x')\right\rangle, \left\langle \psi^\dagger(x)\psi(x')\right\rangle \propto \int dk\, \frac{\cos k(x-x')}{\exp\left(\beta\left(\hbar^2k^2/2m-\mu\right)\right)-1}, \tag{3.36}$$

and, clearly, finite integration in x, x' cannot make expression Eq. (3.36) divergent. However, now the second term that contains the δ-function is also regular:

$$\int_0^l \int_0^l \left\langle \psi^\dagger(x)\psi(x')\right\rangle \delta(x-x')\,dxdx' = l\left\langle \psi^\dagger\psi\right\rangle = \langle p\rangle\,. \tag{3.37}$$

However, the result obtained using classical field approximation, Eq. (3.30), is well defined - it is the square of the ground state wave function in a regular potential. Here we need to remember that the path-integral averages result in the averages of normal-products [73], so the quantity computed in Eq. (3.30) corresponds to

$$w(\rho) = \left\langle :\delta\left(\rho-\psi^\dagger\psi\right):\right\rangle, \tag{3.38}$$

rather than Eq. (3.23), and the moments of Eq. (3.38), instead of $\left\langle \psi^\dagger\psi\psi^\dagger\psi\right\rangle$, $\left\langle \psi^\dagger\psi\psi^\dagger\psi\psi^\dagger\psi\right\rangle$, etc, are $\left\langle \psi^\dagger\psi^\dagger\psi\psi\right\rangle$, $\left\langle \psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi\right\rangle$, etc. And indeed, as was demonstrated from example cal-

culations done for non-interacting bosons Eq. (3.32), while $\left\langle \psi^\dagger(x)\psi(x)\psi^\dagger(x)\psi(x)\right\rangle$ is a singular quantity that contains $\delta(0)$, its normal-ordered version $\left\langle \psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)\right\rangle$ is regular and, using Wick's theorem, evaluates to $2\left\langle \psi^\dagger(x)\psi(x)\right\rangle^2$.

The k-th moment of the particle number distribution on a segment of given length is given by

$$\left\langle p^k\right\rangle = \int_0^l dx_1 dx_2 \dots dx_k \left\langle \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2)\dots\psi^\dagger(x_k)\psi(x_k)\right\rangle. \tag{3.39}$$

We apply the bosonic commutation relations to convert the average in Eq. (3.39) into a sum of normal-ordered averages, along the lines of Eq. (3.32), to obtain

$$\left\langle p^k\right\rangle = \int_0^l dx_1 dx_2 \dots dx_k \left\langle \psi^\dagger(x_1)\psi^\dagger(x_2)\dots\psi^\dagger(x_k)\psi(x_1)\psi(x_2)\dots\psi(x_k)\right\rangle + \dots, \tag{3.40}$$

where ellipses stand for the averages that are lower order in the number of creation/annihilation operators and in the number of integrations. So if the length l of the segment is such that it contains many particles on average,

$$1 \ll l\left\langle \psi^\dagger\psi\right\rangle = l\left\langle \rho\right\rangle, \qquad \frac{1}{\left\langle \rho\right\rangle} \ll l, \tag{3.41}$$

we may assume that the terms omitted in Eq. (3.40) are negligible and write

$$\left\langle p^k\right\rangle \approx \int_0^l dx_1 dx_2 \dots dx_k \left\langle \psi^\dagger(x_1)\psi^\dagger(x_2)\dots\psi^\dagger(x_k)\psi(x_1)\psi(x_2)\dots\psi(x_k)\right\rangle. \tag{3.42}$$

But also, if at the same time pixel size l is smaller than coherence length l_φ,

$$l \ll l_\varphi, \tag{3.43}$$

the correlators $\left\langle \psi^\dagger(x_1)\psi^\dagger(x_2)\dots\psi^\dagger(x_k)\psi(x_1)\psi(x_2)\dots\psi(x_k)\right\rangle$ may be approximated by their values at the origin,

$$\left\langle \psi^\dagger(0)\psi^\dagger(0)\dots\psi^\dagger(0)\psi(0)\psi(0)\dots\psi(0)\right\rangle, \tag{3.44}$$

so that the k-th moment Eq. (3.42) becomes

$$\left\langle p^k\right\rangle \approx \left\langle p\right\rangle^k \frac{\left\langle \psi^\dagger(0)\psi^\dagger(0)\dots\psi^\dagger(0)\psi(0)\psi(0)\dots\psi(0)\right\rangle}{\left\langle \rho\right\rangle^k}. \tag{3.45}$$

Eqs. (3.41) and (3.43) imply that there are many particles inside the correlation length,

$$1 \ll l_\varphi \left\langle \rho\right\rangle, \tag{3.46}$$

which can only hold in the weakly interacting regime.

Notice, that the numerator in Eq. (3.45) is just the k-th moment of the normal ordered distribution function Eq. (3.27), up to the scale factors, see Eq. (3.18). If we consider the distribution of normalized density, $\rho/\left\langle \rho\right\rangle$, obtained from Eq. (3.27), then the fraction

$$\frac{\left\langle \psi^\dagger(0)\psi^\dagger(0)\dots\psi^\dagger(0)\psi(0)\psi(0)\dots\psi(0)\right\rangle}{\left\langle \rho\right\rangle^k} \tag{3.47}$$

is the k-th moment of such a distribution. Such "normalized" distribution can be obtained from $w(\rho)$, Eq. (3.30), by rescaling

$$w(\rho) \to Aw(a\rho), \tag{3.48}$$

where A and a have to be chosen to make the average $\left\langle \rho\right\rangle$ equal to 1 while keeping the distribution normalized to unity.

Figures 3.1 and 3.2 show such normalized distributions obtained for different values of η. It is curious to notice that in the quasicondensate regime $\eta \gg 1$, Fig. 3.2, the particle number

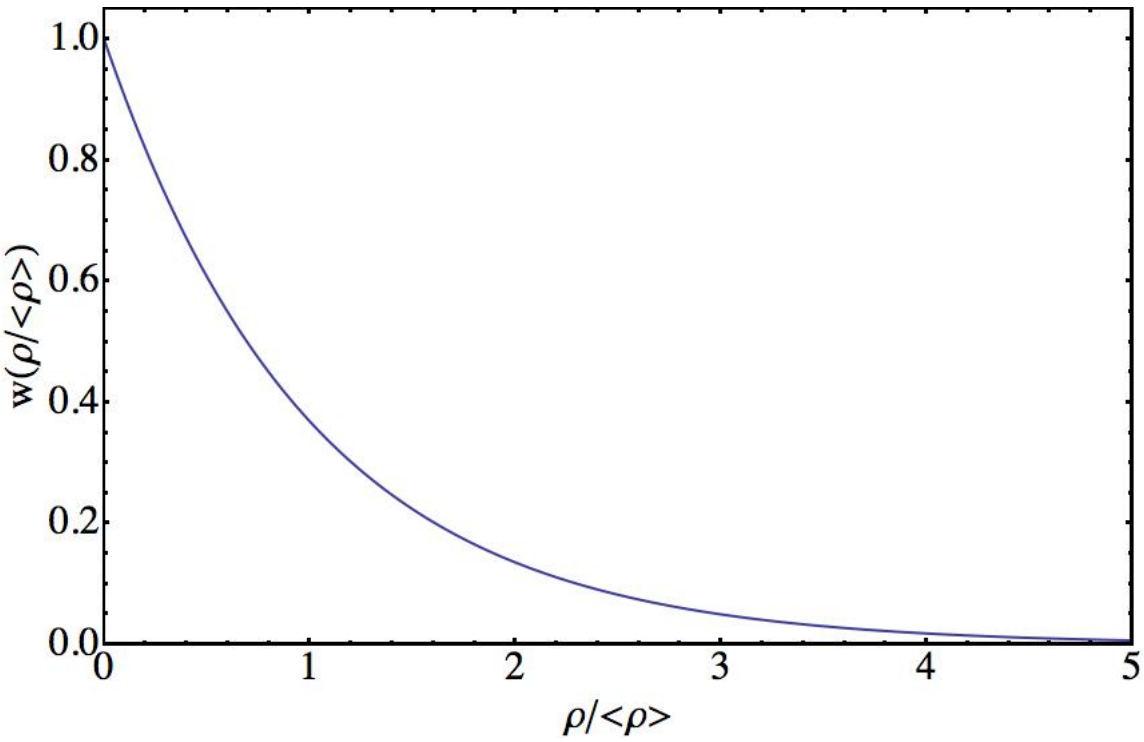


Figure 3.1: Normalized density distribution for $\eta = -\infty$, which corresponds to the strongly degenerate Bose gas limit. The most probable value for density is 0, which is significantly different from its first moment 1, which means that particle number fluctuates strongly from its average value $\langle p \rangle = \rho_0 l$.

distribution is peaked around the average density. This can be related to the fact that in the quasicondensate only the phase fluctuates, whereas the density almost does not. On the other hand in the strongly degenerate gas limit, Fig. 3.1, the most probable particle number is zero. Since the average number of particles is fixed, this implies large fluctuations of the number of particles, of the order of the average number itself.

3.5 Summary

To summarize this Chapter, using the ground state wave functions from the classical field approximation of Chapter 2 we have obtained a full atomic density distribution for a segment

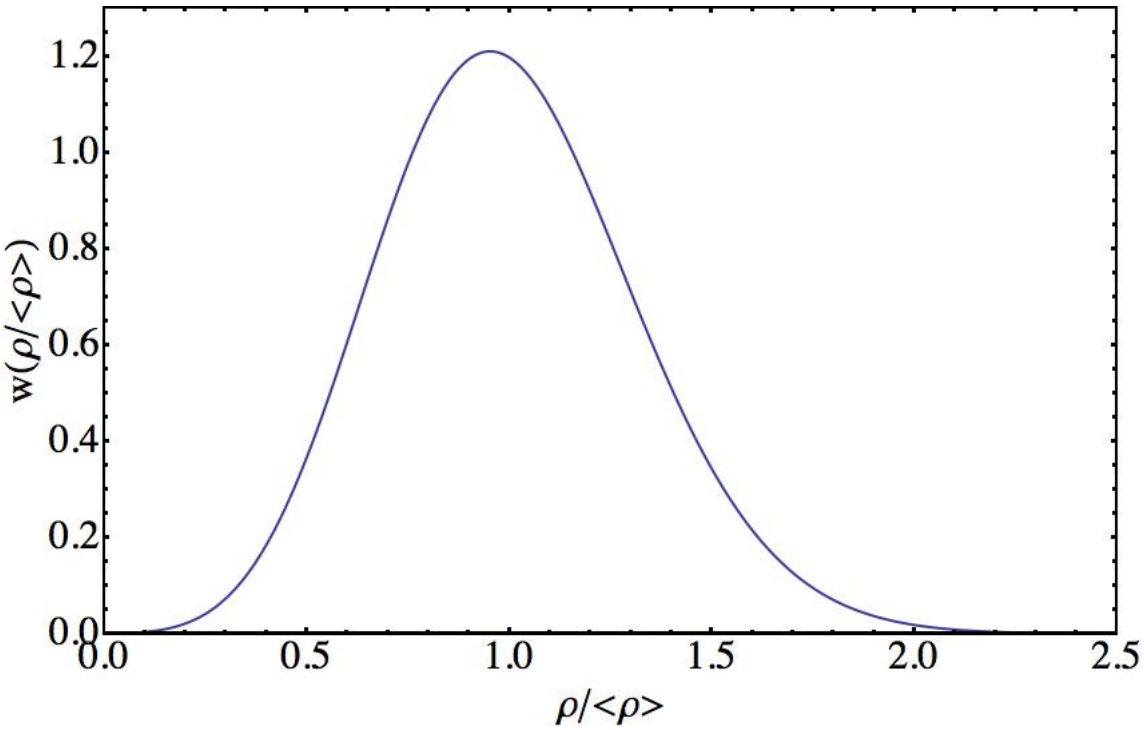


Figure 3.2: Normalized density distribution for $\eta = 4.9$, which is close to the quasicondensate regime $\eta = \infty$. The most probable value for density is close to 1, which means that particle number fluctuates weakly from its average value $\langle p \rangle = \rho_0 l$.

of a weakly interacting 1D system for pixels of certain length scale. Relatively straightforward quantum mechanical calculations thus result in the full density distribution function, meaning that all the moments can be obtained at once.

Chapter 4

Dynamical structure factor

In this Chapter we change focus and, instead of equilibrium momentum correlators, will discuss the dynamics of 1D bosons. Understanding the dynamics of interacting many–body systems, especially at non–zero temperatures, is currently one of the open theoretical problems [45] remaining to be addressed after recent success of the "deep hole" approach towards calculating the density-density correlator in 1D (see [45] and references therein). Namely, the quantity of particular interest to us will be the density–density correlation function

$$S(q,\omega) = \int dxdt\, e^{i(\omega t - qx)} \left(\langle \rho(x,t)\rho(0,0)\rangle - \rho_0^2 \right) = \int dxdt\, e^{i(\omega t - qx)} \langle \delta\rho(x,t)\delta\rho(0,0)\rangle \,, \quad (4.1)$$

also known by the name of the Dynamical Structure Factor (DSF). This correlator is related to the rate of inelastic scattering off the sample [78]. In a qualitative way it is possible to argue that DSF is relevant to dynamics as follows. Rewriting Eq. (4.1) in terms of matrix elements of the density operator ρ in the Schrödinger representation,

$$\delta\rho(x,t) = e^{iHt/\hbar}\delta\rho(x)e^{-iHt/\hbar}, \quad (4.2)$$

$$\langle m|\,\delta\rho(x,t)\,|n\rangle = \langle m|\,\delta\rho(0)\,|n\rangle\, e^{i(E_m-E_n)t/\hbar - i(q_m-q_n)x}, \tag{4.3}$$

$$S(q,\omega) = \sum_{m,n}\int dxdt\, e^{i(\omega t - qx)} e^{-\beta E_m} e^{-i(E_n-E_m)t/\hbar} e^{i(q_n-q_m)x}\,\left|\langle m|\,\delta\rho(0)\,|n\rangle\right|^2, \tag{4.4}$$

$$S(q,\omega) = (2\pi)^2 \sum_{m,n} e^{-\beta E_m}\,\left|\langle m|\,\delta\rho(0)\,|n\rangle\right|^2 \delta\left(\omega - \frac{E_n - E_m}{\hbar}\right)\delta\left(q - (q_n - q_m)\right) \tag{4.5}$$

where spatial homogeneity is assumed, $|m\rangle$, E_m, q_m are the many–body eigenstates and the corresponding energy and momentum eigenvalues, $\rho(0)$ is the Schrödinger operator at spatial position $x = 0$, and canonical ensemble is assumed when averaging. Incidentally, representation of $S(q,\omega)$ as Eq. (4.5) is also a way to see its relation to scattering/decay processes, as the form Eq. (4.5) is related to the decay rate calculated according to Fermi's golden rule. From Eq. (4.5) it is evident that the DSF is dependent on the exact excitation spectrum of the system, and not just on thermodynamic averages. One could speculate that it is possible to construct several model Hamiltonians with similar thermodynamic averages (such as average density, heat capacity, etc.) but with different spectra, and hence with different $S(q,\omega)$. This is exactly the situation we are going to be dealing with in this chapter. Equilibrium properties of interacting 1D bosons were adequately described using the hydrodynamic approach (e.g. see Chapter 2), which turns out to be inadequate for calculating the DSF.

This Chapter follows [79] where M. Arzamasovs has been one of the co-authors. I have done the diagrammatic calculations relevant to finding τ_F in Sections 4.11 and 4.12. I have collaborated with F. Bovo during working on this project.

4.1 Hydrodynamics

Hydrodynamics describes the system in terms of smooth, long wavelength and low energy configurations of density and velocity fields [18, 80]. In fact, we have already used the hydrodynamic

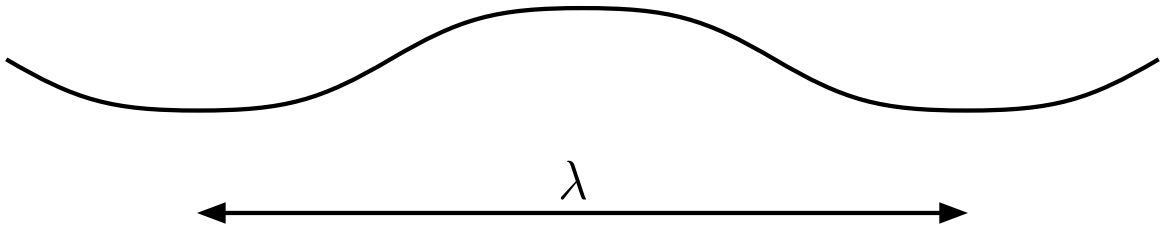


Figure 4.1: Hydrodynamics describes the low energy, long wavelength excitations of wavelength λ, satisfying $2\pi l/\lambda \ll 1$, where l is the mean free path in the underlying microscopic theory [81].

approach to the 1D quantum systems in Chapter 2 to compute the momentum-momentum correlator in the regime of phase-fluctuating (and density-non-fluctuating) quasicondensate. This Chapter is dedicated to investigating whether the hydrodynamic description, such as the Luttinger Liquid model, is also adequate to describe the dynamics, and how it can be modified.

As we have seen in Chapter 2, in 1D it is convenient to use the displacement and phase fields θ and φ, instead of density and velocity. The connection between the two sets of variables is

$$\delta\rho = \frac{1}{\pi}\frac{\partial\theta}{\partial x}, \qquad u = \frac{\hbar}{m}\frac{\partial\varphi}{\partial x}, \tag{4.6}$$

where $\delta\rho(x)$ is the displacement of the density from its average value ρ_0 and $u(x)$ is the superfluid velocity. The low–energy physics can then be described in terms of the variables Eq. (4.6) by the following effective Lagrangian density:

$$\mathcal{L}_{hyd} = -\delta\rho\hbar\frac{\partial\varphi}{\partial t} - \hbar^2\frac{(\rho_0+\delta\rho)}{2m}\left(\frac{\partial\varphi}{\partial x}\right)^2 - \left[e_0(\rho_0+\delta\rho) - e_0(\rho_0) - \mu\delta\rho\right]. \tag{4.7}$$

In Eq. (4.7) the first term is the analogue of the "$p\dot{x}$" term in the one-particle Lagrangian $L = p\dot{x} - H(p,x)$. The second term, "$mu^2/2$", is the kinetic energy and the terms in the square brackets stand for the potential energy, expressed as the difference of local energy densities at

local particle densities $\rho_0 + \delta\rho$ and ρ_0. $\mu = \partial e_0/\partial n$ is the chemical potential.

From here one can continue in the spirit of the Luttinger Liquid model (see Chapter 2), where Eq. (4.7) is expanded around the stationary solution $\rho(x) = u(x) = 0$ up to the second order in $\delta\rho(x)$, $\partial\varphi(x)$ to yield

$$\mathcal{L}_{LL} = -\frac{\hbar}{\pi}\frac{\partial\theta}{\partial x}\frac{\partial\varphi}{\partial t} - -\frac{\hbar c K}{2\pi}\left(\frac{\partial\varphi}{\partial x}\right)^2 - \frac{\hbar c}{2\pi K}\left(\frac{\partial\theta}{\partial x}\right)^2, \tag{4.8}$$

where c is the speed of sound in the system and K is the dimensionless Luttinger parameter, defined by

$$cK = \frac{\hbar\pi n}{m}, \qquad \frac{c}{K} = \frac{1}{\pi\hbar}\frac{\partial\mu}{\partial n}. \tag{4.9}$$

For bosons with weak short–range interactions have $K \gg 1$, while $K \to 1$ corresponds to the Tonks-Girardeau limit of hard core (i.e. infinitely interacting) Bose gas [41], which by virtue of the Jordan-Wigner transform [42] is related to the gas of weakly interacting fermions [42].

The excitations of Eq. (4.8) are phonons with velocity c, which can be highlighted by rewriting the Lagrangian in terms of the chiral fields,

$$\chi_+ = \frac{\theta}{\sqrt{K}} + \sqrt{K}\varphi, \tag{4.10}$$

$$\chi_- = \frac{\theta}{\sqrt{K}} - \sqrt{K}\varphi. \tag{4.11}$$

Then the Lagrangian density Eq. (4.8) becomes

$$\mathcal{L}_{LL} = \frac{\hbar}{4\pi}\chi_+\left(\frac{\partial^2}{\partial x\partial t} + \frac{c\partial^2}{\partial^2 x}\right)\chi_+ + \frac{\hbar}{4\pi}\chi_-\left(-\frac{\partial^2}{\partial x\partial t} + \frac{c\partial^2}{\partial^2 x}\right)\chi_-. \tag{4.12}$$

Solving the relevant Euler-Lagrange equation we obtain the (semi-) classical solutions of Eq.

(4.12):

$$\left(\frac{\partial^2}{\partial x \partial t} + \frac{c\partial^2}{\partial^2 x}\right)\chi_+ = 0, \tag{4.13}$$

$$\left(-\frac{\partial^2}{\partial x \partial t} + \frac{c\partial^2}{\partial^2 x}\right)\chi_- = 0, \tag{4.14}$$

which elucidate the meaning of variables $\chi_\pm$ as the right and left moving phonons with speed c.

4.2 Dynamical structure factor for the Luttinger Liquid

Calculating the DSF corresponding to the Lagrangian Eq. (4.12) is fairly straightforward since the action is quadratic and the density is linear in the field variables:

$$\begin{aligned} S(x,t) &= \frac{1}{\pi^2}\frac{\partial^2}{\partial x_1 \partial x_2}\left\langle \theta(x_1,t)\theta(x_2,0)\right\rangle\big|_{x_1-x_2=x} \\ &= \frac{K}{4\pi^2}\frac{\partial^2}{\partial x_1 \partial x_2}\left(\langle\chi_+(x_1,t)\chi_+(x_2,0)\rangle + \langle\chi_+(x_1,t)\chi_-(x_2,0)\rangle + ,\right. \\ &\quad \left. + \langle\chi_-(x_1,t)\chi_+(x_2,0)\rangle + \langle\chi_-(x_1,t)\chi_-(x_2,0)\rangle\right)\big|_{x_1-x_2=x} \end{aligned} \tag{4.15}$$

where new variables x_1 and x_2 were introduced to aid the transition from $\delta\rho$ to θ. We also made use of spatial homogeneity of the system by setting $x_1 - x_2 = x$ but not fixing the value of each coordinate separately. Since the right and left phonons are independent, the averages containing different types of phonons vanish and the above equation is reduced to

$$S(x,t) = \frac{K}{4\pi^2}\frac{\partial^2}{\partial x_1 \partial x_2}\left(\langle\chi_+(x_1,t)\chi_+(x_2,0)\rangle + \langle\chi_-(x_1,t)\chi_-(x_2,0)\rangle\right)\big|_{x_1-x_2=x}. \tag{4.16}$$

The averages

$$\langle\chi_+(x_1,t)\chi_+(x_2,0)\rangle\,,\ \langle\chi_-(x_1,t)\chi_-(x_2,0)\rangle \tag{4.17}$$

are related to the Green's functions of Eq. (4.12).

Indeed, the retarded Green's function, for both kinds of phonons, is defined as

$$D^R(x_1 - x_2, t_1 - t_2) = -i\left[\langle\chi(x_1,t_1)\chi(x_2,t_2)\rangle - \langle\chi(x_2,t_2)\chi(x_1,t_1)\rangle\right]\Theta(t_1 - t_2) \tag{4.18}$$

where spatial and temporal homogeneity is assumed. Its counterpart, the advanced Green's function is related to D^R through complex conjugation

$$D^A(x_1 - x_2, t_1 - t_2) = \left[D^R(x_2 - x_1, t_2 - t_1)\right]^*. \tag{4.19}$$

For non-interacting phonons, Eq. (4.12), the Green's functions in the Fourier space

$$D(q,\omega) = \int dtdx\, e^{-iqx+i\omega t} D(x,t) \tag{4.20}$$

are known:

$$D^R_{0,+}(q,\omega) = \left[D^A_{0,+}(q,\omega)\right]^* = \frac{\pi}{q(\omega - cq + i\cdot 0)}, \tag{4.21}$$

$$D^R_{0,-}(q,\omega) = \left[D^A_{0,-}(q,\omega)\right]^* = \frac{\pi}{-q(\omega + cq + i\cdot 0)}, \tag{4.22}$$

where the subscript "0" stands for "non–interacting". Eqs. (4.21) and (4.22) can be derived by writing the action for e.g. the right-moving phonons in the Fourier space (using continuous $q-$quantization)

$$\begin{aligned} S_{0,+} &= \int dtdx \left[\frac{\hbar}{4\pi}\chi_+\left(\frac{\partial^2}{\partial x\partial t} + \frac{c\partial^2}{\partial^2 x}\right)\chi_+\right] = \\ &= \int \frac{dqd\omega}{(2\pi)^2}\,\chi(-q,-\omega)\left[\frac{\hbar}{4\pi}q(\omega - cq)\right]\chi(q,\omega), \qquad \chi(-q,-\omega) = \chi^*(q,\omega), \end{aligned} \tag{4.23}$$

and then calculating the path integral averages with this quadratic action,

$$\langle \chi_+(q,\omega)\chi_+(q',\omega')\rangle_0 = \int \mathcal{D}\chi\, \chi_+(q,\omega)\chi_+(q',\omega') \exp\left(\frac{i}{\hbar} S_{0,+}\right), \tag{4.24}$$

to get

$$\langle \chi_+(q,\omega)\chi_+(q',\omega')\rangle_0 = (2\pi)^2\, \delta(q+q')\delta(\omega+\omega') i D_{0,+}^{R/A} =$$

$$= (2\pi)^2\, \delta(q+q')\delta(\omega+\omega') \frac{i\pi}{q(\omega - cq \pm i\cdot 0)}, \tag{4.25}$$

where $+$ or $-$ sign in the denominator is chosen depending on whether we are interested in the retarded or advanced correlator.

In thermal equilibrium at temperature $k_B T = 1/\beta$ there exists a simple relationship [82] between the correlation functions Eq. (4.17) and the Green's functions Eqs. (4.18) and (4.19):

$$\mathrm{Im} D^R(x_1 - x_2, \omega) = -\frac{1}{2}\left(1 - e^{-\beta\omega}\right) J(x_1 - x_2, \omega), \tag{4.26}$$

where $J(x_1 - x_2, \omega)$ is the Fourier transform of (4.17) in time:

$$J(x_1 - x_2, \omega) = \int dt\, e^{i\omega t} \left\langle \chi(x_1, t)\chi(x_2, 0)\right\rangle .$$

Using Eq. (4.19) we rewrite Eq. (4.26) as

$$\frac{1}{2i}\left(D^R(x_1 - x_2, \omega) - D^A(x_2 - x_1, \omega)\right) = -\frac{1}{2}\left(1 - e^{-\beta\omega}\right) J(x_1 - x_2, \omega). \tag{4.27}$$

Taking the spatial Fourier transform of Eq. 4.27 we obtain

$$\frac{1}{2i}\left(D^R(q, \omega) - D^A(-q, \omega)\right) = -\frac{1}{2}\left(1 - e^{-\beta\omega}\right) J(q, \omega). \tag{4.28}$$

The Dynamical Structure Factor in the Fourier space is given by

$$S(q,\omega) = \int dxdt\, e^{-i(qx-\omega t)} S(x,t) = \frac{Kq^2}{4\pi^2}\left(J_+(q,\omega) + J_-(q,\omega)\right), \tag{4.29}$$

where the factor of q^2 comes from the second derivative in Eq. (4.16). Substituting the values of J_+ and J_- Eq. (4.28), for non–interacting case obtain the following expression:

$$S(q,\omega) = \frac{iKq^2}{4\pi^2}\frac{1}{1-e^{-\beta\omega}}\left(D^R_{0,+}(q,\omega) - D^A_{0,+}(-q,\omega) + D^R_{0,-}(q,\omega) - D^A_{0,-}(-q,\omega)\right). \tag{4.30}$$

Due to right-left phonon symmetry, we recognize that $D^A_+(-q,\omega) = D^A_-(q,\omega)$. This yields

$$\begin{aligned} S(q,\omega) &= \frac{iKq^2}{4\pi^2}\frac{1}{1-e^{-\beta\omega}}\left(2i\mathrm{Im}D^R_{0,+}(q,\omega) + 2i\mathrm{Im}D^R_{0,-}(q,\omega)\right) = \\ &= \frac{iKq^2}{4\pi^2}\frac{1}{1-e^{-\beta\omega}}\left(\frac{2i\pi^2}{q}\delta(\omega+cq) - \frac{2i\pi^2}{q}\delta(\omega-cq)\right) \\ &= \frac{K}{2\left(1-e^{-\beta\omega}\right)}\left(q\delta(\omega-cq) - q\delta(\omega+cq)\right). \end{aligned} \tag{4.31}$$

We can understand Eq. (4.31) in qualitative terms. The physical meaning of DSF is that it tells how well the system can absorb excitations of given momentum $\hbar q$ and energy $\hbar\omega$. Since the only excitations of the system are phonons with dispersion

$$\omega = \pm cq \tag{4.32}$$

the corresponding DSF must consist of (weighted) delta–functions centered at those two dispersions.

4.3 Non-linear hydrodynamics

Notice that the arguments of Section 4.2 leading to the result Eq. (4.31) heavily rely on the non–interacting nature of the Lagrangian Eq. (4.12). Indeed, if we also keep the cubic order terms in the expansion of Eq. (4.7), this will add extra terms to the phononic Lagrangian density,

$$\mathcal{L}'_{ph} = \frac{\hbar^2}{12\pi m^*}\left[(\partial_x\chi_+)^3 + (\partial_x\chi_-)^3\right] + \dots , \tag{4.33}$$

which describe collisions between otherwise non-interacting phonons. In principle, terms describing collisions of phonons of different chirality are also present, but have been omitted from Eq. (4.33) under the assumption that at low energies the interaction time between right and left moving phonons is small. Emerging interactions Eq. (4.33) modify the Green's functions and since the interacting action is no longer quadratic in the fields, it is impossible to perform the path integral calculation similar to Eq. (4.24) and obtain closed-form expressions for the interacting $D_+^{R/A}$, $D_-^{R/A}$.

Since the interaction terms Eq. (4.33) are irrelevant from the renormalization group perspective [38], qualitatively we would expect the following changes to the Green's functions Eq. (4.21), and consequently to $S(q,\omega)$:

a) positions of the peaks would shift from their non–interacting values at $\omega = \pm cq$, but only slightly for small q, since we expect the dispersion to be altered by terms of the order of $\Delta\omega(q) \sim \hbar^2 q^2/2m^*$,

b) the imaginary part of the denominator would become finite, reflecting the finite lifetime of now-colliding phonons:

$$D_+^R = \frac{\pi}{q(\omega - cq + i\cdot 0)} \to \frac{\pi}{q(\omega - cq - \Delta\omega_+(q,\omega) + i\Gamma_+(q,\omega))}, \tag{4.34}$$

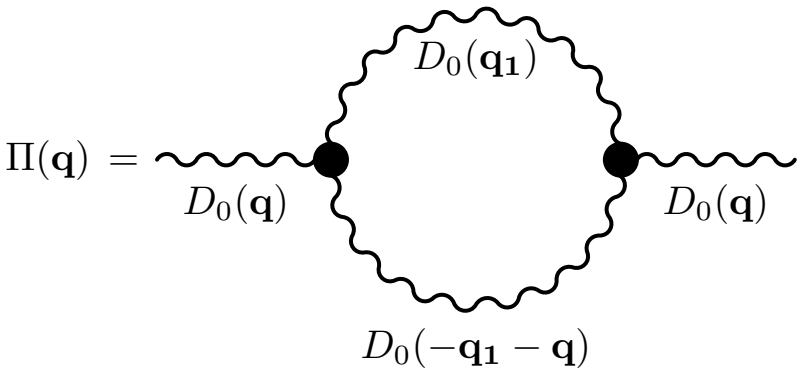


Figure 4.2: Feynman diagram, representing the phononic polarization operator to the lowest order in interaction vertices Eq. (4.33). Boldface $\mathbf{q}$ stands for the pair (q, ω).

and similarly for the left-movers. Here we should recall the Dyson equation,

$$D(q,\omega) = D_0(q,\omega) + D_0(q,\omega)\Pi(q,\omega)D(q,\omega) \quad \longleftrightarrow \quad D^{-1}(q,\omega) = D_0^{-1}(q,\omega) - \Pi(q,\omega), \tag{4.35}$$

where $\Pi(q,\omega)$ is the phononic polarization operator. From the rightmost equation in Eq. (4.35) we identify

$$\frac{q\delta\omega_+(q,\omega)}{\pi} = \mathrm{Re}\Pi_+^R(q,\omega), \qquad \frac{q\Gamma_+(q,\omega)}{\pi} = -\mathrm{Im}\Pi_+^R(q,\omega) \tag{4.36}$$

and similarly for the left-movers. Hence modification to the propagators $D(q,\omega)$ and, consequently, to $S(q,\omega)$ can be accounted for by computing $\Pi_\pm^R(q,\omega)$.

In the spirit of field theory one may anticipate using perturbation theory to calculate the polarization operators up to some order in interaction term Eq. (4.33) to obtain a 1D analogue of the Beliaev damping [29]. But doing the perturbation theory calculations even to the lowest order, see Fig. 4.2, reveals that in 1D the perturbation expansion in Eq. (4.33) is resonant because the conservation momentum of implies conservation of energy (an artifact of linear dispersion in $1D$ for the non-interacting theory). This leads to a divergence when applying

Fermi's Golden rule: the result is multiplied by "$\delta(0)$".

Indeed, let us demonstrate this by writing the action corresponding to Eq. (4.33). In terms of the Fourier components, we obtain

$$\begin{aligned} S'_{ph} &= -\frac{i\hbar^2}{12\pi m^*}\int \frac{dq_1 dq_2 dq_3 d\omega_1 d\omega_2 d\omega_3}{(2\pi)^4} q_1 q_2 q_3 \times \\ &= [\chi_+(q_1,\omega_1)\chi_+(q_2,\omega_2)\chi_+(q_3,\omega_3)\delta(q_1+q_2+q_3)\delta(\omega_1+\omega_2+\omega_3)+\ldots], \end{aligned} \quad (4.37)$$

where the interaction term for the left movers has been left out. For demonstration purposes we state without derivation the expression for Π^R obtained using the Keldysh technique and perturbation theory:

$$\Pi^R(q,\omega) \propto -iq^2 \int dq_1 d\omega_1 \, q_1^2 \, (q-q_1)^2 \left[D^K(q_1,\omega_1)D^R(q-q_1,\omega-\omega_1) + D^R(q_1,\omega_1)D^K(q-q_1,\omega-\omega_1)\right], \quad (4.38)$$

where we do not write the subscript "+" but we assume that Π and D refer to the right-moving phonons. D^K is the Keldysh component of the propagator, see Section 4.9 for definitions. For the purposes of this section it is enough to set

$$D^K(q,\omega) = 2i \coth\left(\frac{\hbar\omega}{2k_BT}\right) \mathrm{Im}D^R. \quad (4.39)$$

For the non-interacting right phonons, D_0^R is given by Eq. (4.21) and the Keldysh component D_0^K is

$$D_0^K(q,\omega) \propto \frac{i}{q}\delta(\omega-cq)\coth\left(\frac{\hbar cq}{2k_BT}\right). \quad (4.40)$$

By substituting the non-interacting values of D_0^R and D_0^K into the expression for $\Pi^R(q,\omega)$ Eq. (4.38) the following expression is obtained:

$$\Pi^R(q,\omega) \propto q^2 \frac{\int dq_1\, q_1(q-q_1)\left[\coth\left(c\hbar(q_1-q)/k_BT\right)-\coth\left(c\hbar q_1/k_BT\right)\right]}{\omega - cq + i\cdot 0}. \tag{4.41}$$

By extracting the real and imaginary parts from Eq. (4.41) we obtain, assuming that the dispersion renormalisation is insignificant [43, 83], that the decay rate of interacting phonons, $\Gamma_+(q,cq) = -\pi \mathrm{Im}\Pi^R(q,cq)/q$, is indeed proportional to "$\delta(0)$". This result is telling of the fact that perturbation theory up to the finite order is not sufficient for this problem.

The problem of computing $\Pi(q,\omega)$ in such a situation was first addressed by Andreev [43] in the context of general 1D linear hydrodynamics and was later applied by Samokhin [83] specifically to the Luttinger Liquid model. Andreev's approach is instead of computing $\Pi^{R/A}_{\pm}(q,\omega)$ perturbatively to rewrite Eq. (4.38) as a self-consistent equation which $\Pi^R(q,\omega)$ must satisfy,

$$\begin{aligned}\Pi^R(q,\omega) \;\propto\; & q^2 \int dq_1 d\omega_1\, q_1\,(q-q_1)\times \\ & \times\left[\coth\left(c\hbar q_1/k_BT\right)\left(\mathrm{Im}\frac{1}{\omega_1 - cq_1 - \Pi^R(q_1,\omega_1)}\right)\frac{1}{\omega-\omega_1-cq+cq_1-\Pi^R(q-q_1,\omega-\omega_1)}+\right. \\ & \left.+\coth\left(c\hbar(q-q_1)/k_BT\right)\frac{1}{\omega_1 - cq_1 - \Pi^R(q_1,\omega_1)}\left(\mathrm{Im}\frac{1}{\omega-\omega_1-cq+cq_1-\Pi^R(q-q_1,\omega-\omega_1)}\right)\right].\end{aligned} \tag{4.42}$$

The self-consistent approximation also assumes that $\Pi^R_+(q,\omega)$ is

a) purely imaginary, so that dispersion renormalization $\Delta\omega_+$ is insignificant,

b) centered on the line $\omega = cq$ and that

c) $\Pi^R_+(q,\omega) = \gamma_q f\left((\omega-cq)/\gamma_q\right)$ where $f(x)$ is a universal function of width 1, peaked at $x = 0$ with $\max f(x) \sim O(1)$ and rapidly decaying away from the maximum (see [83] for details), and similarly for $\Pi^R_-(q,\omega)$. One then obtains [43, 83] that $S(q,\omega)$ still has two peaks

centered at $\omega = \pm cq$ but the δ-functions acquire finite width, proportional to $\gamma_q(T)$,

$$\gamma_q(T=0) \propto q^2, \qquad \gamma_q(T=0) \propto \sqrt{\frac{k_B T}{\hbar c}} q^{3/2}. \tag{4.43}$$

In Eq. (4.43) we have omitted the common prefactor with dimensions 1/mass and any numerical prefactors, that are in general not given by the self–consistent treatment.

In the end we have obtained a result for the DSF of the interacting $1D$ system, which takes into account the non-linear dispersion of constituent particles (remember, that nonlinearity caused the interaction term Eq. (4.33) in the first place). This result is supposed to hold for arbitrary values of K. In the next section we check Eq. (4.43) against the exactly solvable model of free fermions, which should be described by the Luttinger Liquid model with $K = 1$.

4.4 Free fermions with quadratic spectrum

Indeed, the result Eq. (4.43) is supposed to hold for any value of Luttinger parameter K, in particular for $K = 1$. As was already mentioned above, this is the Tonks-Girardeau gas, corresponding to impenetrable bosons, which can be exactly mapped on the free fermions [41]. This identification allows to check the results of Andreev's theory for the particular case $K = 1$.

In fact the density–density correlator of free (spinless) fermions with quadratic dispersion

$$H = \sum_k \left(\frac{\hbar^2 k^2}{2m} \right) c_k^\dagger c_k \tag{4.44}$$

and at temperature T, $\rho_0 = N/L$ and chemical potential μ, can be calculated exactly. By performing trivial but tedious calculations using fermionic anti-commutation relations and Wick's

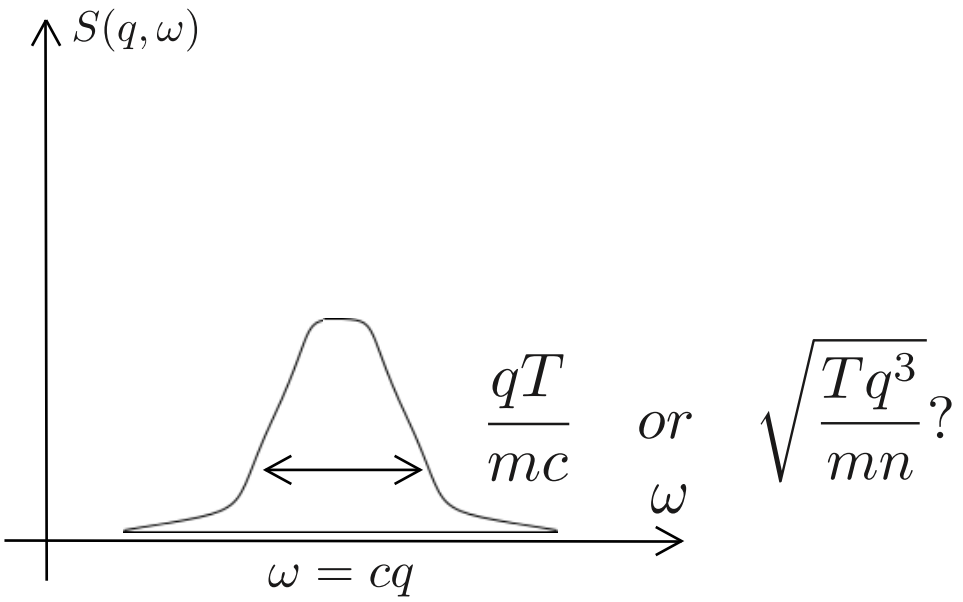


Figure 4.3: The positive frequency peak in DSF of Tonks-Girardeau gas ($K = 1$). From free fermionic calculations of this section it is evident that the peak is formed as an overlap of two Fermi functions. Therefore, at certain temperatures $T > 0$ the width of the peak is expected to scale linearly with q. On the other hand, self–consistent hydrodynamic results in a $q^{3/2}$ scaling law. Which result should we trust at arbitrary K?

theorem, we obtain the following result (for detailed derivation see Appendix A):

$$S(q,\omega) = \frac{m}{\hbar q} n_F\left(E_{\frac{m\omega}{\hbar q}+\frac{q}{2}}\right)\left(1 - n_F(E_{\frac{m\omega}{\hbar q}-\frac{q}{2}})\right). \tag{4.45}$$

We notice that Eq. (4.45) is quite at odds with hydrodynamic result Eq. (4.43) at $T > 0$, see Fig. 4.3.

To summarize so far, in Sections 4.2, 4.3 and 4.4 it was shown that including non-linear terms in the Luttinger Liquid action cannot consistently account for the broadening of the Dynamical Structure Factor. Andreev's approach (Section 4.3) might work for the weakly interacting Bose gas, as was numerically demonstrated in [44], but when used near the Tonks-Girardeau (TG)

limit ($K \to 1$, same as the free fermions) it leads to an inconsistency: the width of the peak of $S(q, \omega)$ scales as $\delta\omega \propto q^{3/2}$ according to the self-consistent hydrodynamics, but the exact calculation gives linear scaling, $\delta\omega \propto q$.

Our solution to this discrepancy is reported in a recently published paper [79] and its outline is the following. In our opinion, the problem with the hydrodynamic approach of Section 4.3 is that from the start we restrict excitations to be phonons, i.e. smooth, long-wavelength waves, which in the case of TG regime is just not true. On the contrary, in the TG gas we expect the excitations to be weakly interacting particle-hole pairs, similar to the case of free fermions. Particle-hole-like excitations are also obtained as the solutions to the Lieb-Liniger model [31, 32], for any (positive) value of the interaction parameter g (see Eq. (2.3)). So, it seems that by "collectivizing" the particles and holes into the hydrodynamic variables some essential physics is lost. The idea that proper excitation picture, necessary to study dynamics, can be restored by adding back the possibility for a system to produce sharp, particle-hole-like excitations in addition to phonons was explored by Pustilnik, Kamenev, Glazman etc. (see [45] and references therein). There it was shown how $S(q, \omega)$ at $T = 0$ can be found by allowing one such hole to be considered separately from the linear phonons. In [79] we adopt this idea to calculate $S(q, \omega)$ at finite temperature, where many such sharp excitations should be taken into account.

4.5 Linear hydrodynamics + impurity

A convenient toolkit for working with sharp excitations, residing on top of otherwise smooth fields of linear hydrodynamics was proposed in Schecter, Gangardt and Kamenev's treatment of a mobile impurity in a superfluid background [84]. The impurity, a foreign object moving through the superfluid, is dragging with it the depletion cloud. Since the impurity is already

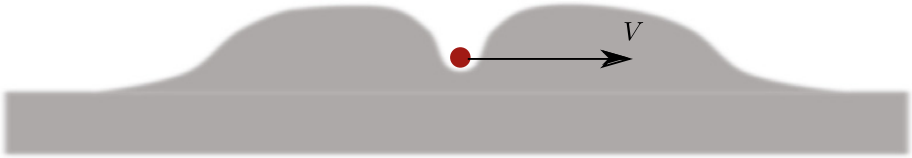


Figure 4.4: Figure taken from [84] that illustrates the phonons in the superfluid background (two large bulges) and a point-like impurity, that also creates a depletion of the fluid around it. If the impurity has mass $M = 0$, then we have a model where phonons coexist with a sharp excitation, e.g. a hole, a soliton, etc.

assumed to be a point object, by also taking its mass $M \to 0$, the remaining depletion cloud can be treated as a sharp excitation of the system itself. We will call such sharp configuration a depleton, an impurity or a hole, interchangeably.

In [84] it was found that the point–like impurity can be described by the following Lagrangian:

$$L = P\dot{X} - H(P, \Lambda) + \Lambda^\dagger \frac{d}{dt}\chi(X, t) + L_{ph}. \tag{4.46}$$

In Eq. (4.46) $L_{ph} = \int dx\, \mathcal{L}_{LL}$ is the linear phonon action Eq. (4.12), $\chi^\dagger = (\chi_+, \chi_-)$ describe phonons, as before, X and P are the coordinate and momentum of the point-like impurity, and variables $\Lambda^\dagger = (\Lambda_+, \Lambda_-)$ describe the depletion cloud around the impurity. In fact, (Λ_+, Λ_-) is just a convenient combination of other depletion cloud variables,

$$\Lambda_+ = \frac{N}{\sqrt{K}} + \frac{\Phi}{\pi}\sqrt{K}, \tag{4.47}$$

$$\Lambda_- = \frac{N}{\sqrt{K}} - \frac{\Phi}{\pi}\sqrt{K}, \tag{4.48}$$

that have more obvious interpretation: in Eqs. (4.47) and (4.48) N and Φ are the drops in the superfluid particle number and the phase across the depletion cloud created by the impurity, see Fig. 4.4.

Eq. (4.46) tells us that there is a convenient length-scale separation: point impurity only interacts with the depletion cloud around it, but does not directly interact with the long wavelength phonons. This is reflected in the first two terms, $P\dot{X} - H(P, \Lambda)$, which describe the dynamics of the impurity at given values of the depletion cloud variables. In the absence of phonons, $\partial\chi_{\pm}/\partial x$, and with impurity mass $M = 0$, N and Φ take their equilibrium values N^* and Φ^*, defined by Eqs. (21) of [84]:

$$P = n\Phi^* - mN^* v(P), \tag{4.49}$$

$$\frac{\partial E(P,n)}{\partial n} = \frac{mc^2}{n} N^* - v(P)\Phi^*. \tag{4.50}$$

Here $E(P, n)$ is the dispersion of the impurity in the absence of phonons,

$$H(P, \Lambda^*) = E(P), \tag{4.51}$$

and $v(P)$ is velocity of the impurity,

$$v(P) = \frac{\partial E(P,n)}{\partial P}. \tag{4.52}$$

Phonons, in their turn, only "see" the depletion cloud, but not the "impurity" itself, which is reflected in the presence of the term $\Lambda^\dagger d\chi(X,t)/dt$, but absence of any coupling between Λ and (X, P).

To reiterate, the mobile impurity approach of [84] works equally well in the case of a foreign particles embedded in the superfluid, such as an atom in a different hyperfine state or a ^3He atom in ^4He superfluid background [84, 85, 86], as well as in the case of localized excitations of the superfluid itself (i.e. no foreign object present), such as solitons in the BEC [87, 88]. It

is the latter case we are interested in: the sharp, non–smooth configurations of the collective fields.

Having the single impurity picture available, the question we eventually need to address is: how the possibility of exciting the "particle-hole pairs", in addition to phonons, will change the DSF. An essential difference at $T > 0$ from the zero-temperature result described in [45] is that the possibility of exciting not just one, but many particle-hole pairs, must be allowed. As in the elementary examples of e.g. free bosons or fermions [73], the most convenient path to many–particle description is second quantization of a single particle Lagrangian. In order to second quantize Eq. (4.46), the following preliminary steps need to be taken.

First of all we integrate out the depletion cloud variables (Λ_+, Λ_-) so that only the phonons and the impurity are left, interacting through an emergent interaction term. Integrating (Λ_+, Λ_-) is a formidable task in general, since the exact form of $H(P, \Lambda)$ is not known analytically, and even if it is, the path integral

$$e^{iS_{eff}} = \int \mathcal{D}\Lambda_+ \mathcal{D}\Lambda_- e^{iS} \tag{4.53}$$

cannot be computed in general. However, if we may assume smallness of the phonon background $(\chi_+, \chi_-) \sim 0$ (at $k_B T/mc^2 \ll 1$), then $H(P, \Lambda)$ can be expanded around flat background,

$$H(P, \Lambda^* + \delta\Lambda) \approx H(P, \Lambda^*) + \delta\Lambda^\dagger \left(2\pi\Gamma_P^{-1}\right) \delta\Lambda = E(P) + \delta\Lambda^\dagger \left(2\pi\Gamma_P^{-1}\right) \delta\Lambda, \tag{4.54}$$

yielding the action quadratic in $\delta\Lambda$:

$$S \approx \int dt\, P\dot{X} - E(P) + \Lambda^{*\dagger} \frac{d}{dt}\chi(X,t) + \delta\Lambda^\dagger \frac{d}{dt}\chi(X,t) - \delta\Lambda^\dagger \left(2\pi\Gamma_P^{-1}\right) \delta\Lambda + L_{ph}. \tag{4.55}$$

In Eqs. (4.54) and (4.55) Γ_P^{-1} is the matrix of second derivatives of $H(P,\Lambda)$,

$$\Gamma_P^{-1} = \frac{1}{4\pi}\begin{pmatrix} \partial^2 H(P,\Lambda)/\partial\Lambda_+\partial\Lambda_+ & \partial^2 H(P,\Lambda)/\partial\Lambda_+\partial\Lambda_- \\ \partial^2 H(P,\Lambda)/\partial\Lambda_-\partial\Lambda_+ & \partial^2 H(P,\Lambda)/\partial\Lambda_-\partial\Lambda_- \end{pmatrix}_{\Lambda=\Lambda^*} \tag{4.56}$$

taken at the equilibrium value $\Lambda = \Lambda^*$. Then $\delta\Lambda$ can be integrated out to yield the effective Lagrangian

$$L = L_{eff} = P\dot{X} - E(P) + \Lambda^{*\dagger}\frac{d}{dt}\chi(X,t) + \frac{1}{8\pi}\left(\frac{d}{dt}\chi^\dagger(X,t)\right)\Gamma_P\left(\frac{d}{dt}\chi(X,t)\right) + L_{ph}. \tag{4.57}$$

Second step is to get rid of the time derivatives in Eq. (4.57). We would like to avoid them, since second-quantizing them is not trivial. To do this, we first express the full time derivative of the phonon fields, evaluated at the impurity position, as

$$\frac{d}{dt}\begin{pmatrix} \chi_+(X,t) \\ \chi_-(X,t) \end{pmatrix} \approx \left(\frac{\partial}{\partial t} + v_+(P)\frac{\partial}{\partial x}\right)\begin{pmatrix} \chi_+(X,t) \\ \chi_-(X,t) \end{pmatrix}, \tag{4.58}$$

where we have used the semiclassical equation of motion for the (right-moving) impurity

$$\frac{\partial X}{\partial t} = v_+(P) = \frac{\partial E}{\partial P} = c + \frac{P}{m^*}, \tag{4.59}$$

$$E_+(P) = cP + \frac{P^2}{2m^*}. \tag{4.60}$$

The quadratic dispersion is used for the impurity to be consistent with the Lieb-Liniger model at low momenta, $P < P_G = mc/\sqrt{K}$ [43]. We then use the semiclassical equations of motion for phonons,

$$\left(\frac{\partial}{\partial t} \pm c\frac{\partial}{\partial x}\right)\chi_\pm = 0, \tag{4.61}$$

to get rid of the partial time derivative, too:

$$\frac{d}{dt}\begin{pmatrix}\chi_+(X,t)\\ \chi_-(X,t)\end{pmatrix} \approx \left(\frac{\partial}{\partial t}+v_+(P)\frac{\partial}{\partial x}\right)\begin{pmatrix}\chi_+(X,t)\\ \chi_-(X,t)\end{pmatrix} \approx \begin{pmatrix}(-c+v_+(P))\,\partial\chi_+(X,t)/\partial x\\ (c+v_+(P))\,\partial\chi_-(X,t)/\partial x\end{pmatrix}. \tag{4.62}$$

Applying Eq. (4.62) to the interaction term of Eq. (4.57) we obtain

$$L_{ph-F} \approx \frac{1}{8\pi}\begin{pmatrix}(-c+v_+(P))\,\partial\chi_+(X,t)/\partial x\\ (c+v_+(P))\,\partial\chi_-(X,t)/\partial x\end{pmatrix}^{\dagger} \Gamma_P \begin{pmatrix}(-c+v_+(P))\,\partial\chi_+(X,t)/\partial x\\ (c+v_+(P))\,\partial\chi_-(X,t)/\partial x\end{pmatrix}. \tag{4.63}$$

Since we are interested in low temperatures, $k_BT \ll \mu = mc^2$, so that quadratic dispersion Eq. (4.59) is at all applicable, we may keep only the lowest order in P terms in Eq. (4.63), without loss of important physics:

$$L_{ph-F} \approx \frac{1}{8\pi}\begin{pmatrix}(P/m^*)\,\partial\chi_+(X,t)/\partial x\\ 2c\partial\chi_-(X,t)/\partial x\end{pmatrix}^{\dagger} \begin{pmatrix}\Gamma_{P,++} & \Gamma_{P,+-}\\ \Gamma_{P,+-} & \Gamma_{P,--}\end{pmatrix} \begin{pmatrix}(P/m^*)\,\partial\chi_+(X,t)/\partial x\\ 2c\partial\chi_-(X,t)/\partial x\end{pmatrix}, \tag{4.64}$$

$$\begin{aligned} L_{ph-F} \;\approx\; & \frac{c^2}{2\pi}\left(\frac{\partial\chi_-(X,t)}{\partial x}\right)\left(\frac{\partial\chi_-(X,t)}{\partial x}\right)\Gamma_{P,--} + \\ & + \frac{1}{2\pi}\frac{c}{m^*}\left(\frac{\partial\chi_+(X,t)}{\partial x}\right)\left(\frac{\partial\chi_-(X,t)}{\partial x}\right)\Gamma_{P,+-}P + \Gamma_{P,++}O(P^2), \end{aligned} \tag{4.65}$$

where we use the symmetry of the matrix Γ_P and do not distinguish between $\Gamma_{P,+-}$ and $\Gamma_{P,-+}$. The first term in the expansion Eq. (4.65) does not contribute since the corresponding process is forbidden by the energy-momentum conservation, see Fig. 4.5. We will show in Section 4.6 that $\Gamma_{P,+-} \approx \Gamma_0' P$ at small momenta, $P < mc$, so that the final expression for the impurity-phonons

interaction term becomes

$$L_{ph-F} \approx \frac{\Gamma_0'}{2\pi}\frac{c}{m^*}\left(\frac{\partial\chi_+(X,t)}{\partial x}\right)\left(\frac{\partial\chi_-(X,t)}{\partial x}\right)P^2, \tag{4.66}$$

which contains only spatial derivatives taken at the position of impurity. Eq. (4.66) therefore can be second–quantized as a position-dependent potential. The only possible ambiguity lies in the ordering of χ-s and P-s, and this will be discussed in Section 4.7.

For left-moving impurities, $P < 0$, analogous calculations can be performed, with the only difference being the change of dispersion relation,

$$E_-(P) = -cP + \frac{P^2}{2m^*}. \tag{4.67}$$

4.6 Backscattering amplitude Γ_P

An essential component of the effective single-impurity Lagrangian Eq. (4.57) is the two-by-two matrix Γ_P, Eq. (4.56). Just like the dispersion $E(P)$, it needs to be specified to be able to describe the dynamics of the impurity. Fortunately, in a saddle-point approximation that leads to Eq. (4.57) we are only interested in Γ_P evaluated at the equilibrium value of Λ, $\Lambda = \Lambda^*$. We have seen in Section 4.5 that the only component of matrix Γ_P that needs to be accounted for is $\Gamma_{P,-+} = \Gamma_{P,+-}$, describing the rate of backscattering of right-moving phonons to the left-moving and vice versa. Two other components, $\Gamma_{P,++}$ and $\Gamma_{P,--}$, either only matter at higher momenta or are irrelevant due to restrictions imposed by the energy-momentum conservation laws.

The expression for calculating the quantity $\Gamma_{P,+-}$ at equilibrium values of $\Lambda_+ = \Lambda_+^*$, $\Lambda_- =$

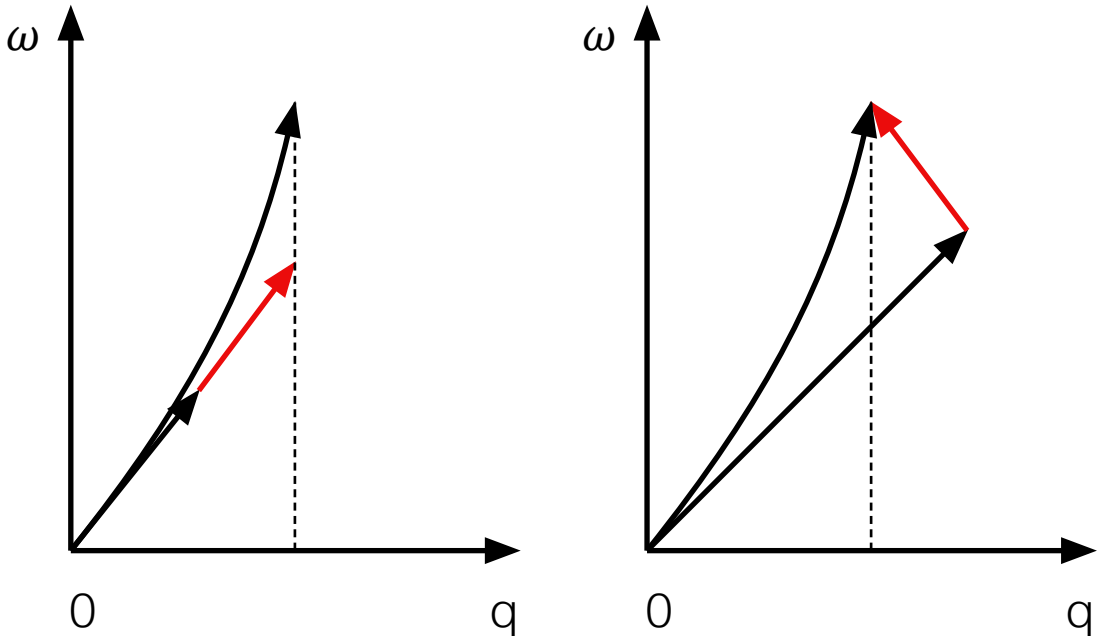


Figure 4.5: Two right phonons with linear dispersion cannot combine to give an impurity with quadratic dispersion (picture on the left), while conserving both momentum and energy, neither can two left phonons. On the other hand a right and a left phonon can (picture on the right).

Λ^*_- (or $N = N^*$, $\Phi = \Phi^*$) was given in [84]. Namely, Eq. (75) of [84] is

$$\Gamma(P,n) = \Gamma_{P,+-} = -\frac{1}{c}\left(\frac{M}{m}\frac{\partial\Phi^*}{\partial P} + \Phi^*\frac{\partial N^*}{\partial P} - N^*\frac{\partial\Phi^*}{\partial P} + \frac{\partial N^*}{\partial n}\right), \tag{4.68}$$

where n is the background density of the liquid, m is the bare mass of the particles, comprising the liquid, and M would be the mass of foreign object forming the impurity. Since in our case the "impurity" is just a sharp configuration of the fields, an excitation of the liquid itself, we put $M = 0$ and the first term in Eq. (4.68) simply gets crossed out.

Further, Eq. (21) of [84] gives the expressions for the equilibrium values of the depletion cloud variables, N^* and Φ^*:

$$v(P,n) = \frac{\partial E(P,n)}{\partial P} = \frac{P - n\Phi^*}{M - mN^*} = \frac{n\Phi^* - P}{mN^*}, \qquad \frac{\partial E(P,n)}{\partial n} = \frac{mc^2}{n}N^* - v(P,n)\Phi^*, \tag{4.69}$$

where again we need to set $M = 0$. From now on we drop asterisks on the equilibrium values,

$$N^* \to N, \qquad \Phi^* \to \Phi, \tag{4.70}$$

and investigate the small–momentum behavior of N, Φ and Γ_P:

$$N(P) = N_0 + N_1 P + N_2 P^2 + \dots, \tag{4.71}$$

$$\Phi(P) = \Phi_0 + \Phi_1 P + \Phi_2 P^2 + \dots, \tag{4.72}$$

$$\mathit{\Gamma}(P) = \mathit{\Gamma}_0 + \mathit{\Gamma}'_0 P + \dots. \tag{4.73}$$

Using Eq. (4.68) we can express Γ_0 and Γ_0' in terms of N, Φ:

$$\Gamma_0 = -\frac{1}{c}\left(\Phi_0 N_1 - N_0\Phi_1 + \frac{\partial N_0}{\partial n}\right) \tag{4.74}$$

and

$$\Gamma_0' = -\frac{1}{c}\left(2\Phi_0 N_2 - 2N_0\Phi_2 + \frac{\partial N_1}{\partial n}\right). \tag{4.75}$$

By inverting Equations (4.69) we obtain full equilibrium $N(P)$, $\Phi(P)$

$$N(P) = \frac{1}{m}\frac{v(P,n)P + n\partial E(P,n)/\partial n}{c^2 - v^2(P,n)}, \qquad \Phi(P) = \frac{P}{n} + \frac{mv(P,n)}{n}N(P), \tag{4.76}$$

that we now need to expand in order to obtain the Taylor coefficients N_i, Φ_i.

As was briefly mentioned in Section 4.5, we use a quadratic dispersion

$$E(P,n) = cP + \frac{P^2}{2m^*} \tag{4.77}$$

since that is the form the dispersion takes in the Lieb-Liniger model for $P < P_G = mc/\sqrt{K}$ [89]. Of course, at higher momenta the dispersion becomes consistent with the Bogoliubov theory, i.e. cubic in P. We are currently interested in small momenta, so we limit ourselves to $P < P_G$ and consider $P > P_G$ separately.

With the choice Eq. (4.77), derivatives of the dispersion become

$$\frac{\partial E(P,n)}{\partial P} = v(P,n) = c + \frac{P}{m^*}, \qquad \frac{\partial E(P,n)}{\partial n} = \frac{\partial c}{\partial n}P - \frac{P^2}{2(m^*)^2}\frac{\partial m^*}{\partial n}. \tag{4.78}$$

In addition we use the expression for the effective mass m^* in the Luttinger Liquid cited in

[45, 90]

$$\frac{1}{m^*} = \frac{c}{K}\frac{\partial}{\partial\mu}\left(c\sqrt{K}\right) = \frac{1}{2m\sqrt{\frac{\pi n}{mc}}}\left(1 + \frac{n}{c}\frac{\partial c}{\partial n}\right). \tag{4.79}$$

By finally putting all the pieces together, we obtain the zeroth order coefficients

$$N_0 = -\frac{m^*}{2m}\left(1 + \frac{n}{c}\frac{\partial c}{\partial n}\right) = -\sqrt{K}, \tag{4.80}$$

$$\Phi_0 = -\frac{m^* c}{2n}\left(1 + \frac{n}{c}\frac{\partial c}{\partial n}\right) = -\frac{\pi}{\sqrt{K}}, \tag{4.81}$$

and the first order coefficients

$$N_1 = \frac{1}{2mc}\left(\frac{n}{2m^*}\frac{\partial m^*}{\partial n} + \frac{n}{2c}\frac{\partial c}{\partial n} - \frac{1}{2}\right), \tag{4.82}$$

$$\Phi_1 = \frac{1}{2n}\left(\frac{n}{2m^*}\frac{\partial m^*}{\partial n} - \frac{n}{2c}\frac{\partial c}{\partial n} + \frac{1}{2}\right). \tag{4.83}$$

Plugging Eqs. (4.80), (4.81), (4.82) and (4.83) into Eq. (4.74), Γ_0 simplifies to

$$\mathit{\Gamma}_0 = -\frac{m^*}{4mcn}\left(1 - \left(\frac{n}{c}\frac{\partial c}{\partial n}\right)^2\right) - \frac{1}{c}\frac{\partial N_0}{\partial n}. \tag{4.84}$$

Substituting Eq. (4.79) for m^* and using

$$-\frac{\partial N_0}{\partial n} = \frac{\partial}{\partial n}\sqrt{\frac{\pi n}{mc}} = \sqrt{\frac{\pi}{mnc}}\left(1 - \frac{n}{c}\frac{\partial c}{\partial n}\right), \tag{4.85}$$

we find that

$$\mathit{\Gamma}_0 = 0, \tag{4.86}$$

i.e. that it vanishes identically. This means that at low P the expansion of $\Gamma(P)$ starts directly from the first-order term.

Γ_0' takes the form

$$\Gamma_0' = -\frac{1}{c}\left(\left(1+\frac{n}{c}\frac{\partial c}{\partial n}\right)\frac{N_1}{n}+\frac{\partial N_1}{\partial n}\right), \tag{4.87}$$

since the terms containing N_2 and Φ_2 cancel because of the following relation:

$$\Phi_2 = \frac{mc}{n}N_2 + \frac{m}{m^*n}N_1. \tag{4.88}$$

For

$$\mu(n) = gn, \qquad c(n) = \sqrt{\frac{gn}{m}}, \tag{4.89}$$

the following result for Γ_0' is obtained:

$$\Gamma_0' = \frac{1}{16mc^2n}. \tag{4.90}$$

We, however, expect $\Gamma(P)$ to vanish identically for integrable systems, such as the Lieb-Liniger model – we don't expect any backscattering if the impurity represents an eigenstate of the system. The non-zero result of Eq. (4.90) is really an artifact of the mean-field approximation Eq. (4.89) used for $\mu(n)$. On the other hand, the contribution due to the three-body collisions, for which the mean-field chemical potential and the speed of sound are

$$\mu(n,\alpha) = gn + \frac{\alpha}{2}n^2, \qquad c(n,\alpha) = \sqrt{\frac{gn+\alpha n^2}{m}} \approx \sqrt{\frac{gn}{m}}\left(1+\frac{\alpha n}{2g}\right), \tag{4.91}$$

is not expected to vanish. Using the modified expression Eq. (4.91) for the velocity of sound in Eq. (4.87) and subtracting Eq. (4.90), we obtain

$$\Gamma_0' \approx -\frac{1}{48}\frac{\alpha n}{m^2c^4}, \qquad \Gamma(P) = \Gamma_0' P, \tag{4.92}$$

up to the first order in α.

4.7 Field theory: Second Quantization

If our aim is describing the 1D dynamics at non–zero temperatures, we have to extend the theory to allow an arbitrary number of such impurities. Assuming that the "gas" of impurities is dilute enough to exclude any direct impurity-impurity interactions, we will now show how we can second-quantize the effective single impurity Lagrangian Eq. (4.57).

The recipe [73] is to introduce the right (+) and left (-) moving impurity creation and annihilation operators $\bar{\psi}_{\pm}, \psi_{\pm}$, and to "sandwich" every term of Lagrangian Eq. (4.57) that contains either P or X between a creation and annihilation operator:

$$L(X, P) \to \int dx\, \bar{\psi} L(x, P) \psi. \tag{4.93}$$

The single impurity momentum P is promoted to derivative operator:

$$P \to -i\hbar \frac{\partial}{\partial x}. \tag{4.94}$$

and quite separately from the rules laid above, the "Berry phase" term $P\dot{X}$ goes to

$$P\dot{X} \to \int dx\, i\hbar \bar{\psi} \frac{\partial \psi}{\partial t}. \tag{4.95}$$

We use fermionic quantization for the impurities for several reasons. One is again the Tonks-Girardeau regime, $K \to 1$, in which the impurities are exactly the fermionic particle-hole pairs. Another reason is a sort of Pauli exclusion principle that holds for impurities: two or more impurities cannot reside at the same point in space, see Fig. 4.4, this will rather be another

impurity with different N and Φ.

Following the rules Eqs. (4.93), (4.94) and (4.95) outlined above one can quite easily second-quantize the dispersion term of Eq. (4.57). For example, for the right-movers we have

$$P\dot{X} - E_+(P) = P\dot{X} - cP + \frac{P^2}{2m^*} \to \int dx\, \bar{\psi} \left(i\hbar \frac{\partial}{\partial t} + ic\hbar \frac{\partial}{\partial x} + \frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial x^2} \right) \psi. \tag{4.96}$$

On the other hand the phononic Lagrangian L_{ph} does not explicitly depend on the impurity variables, so it remains unaltered. In order to second-quantize the phonon-impurity interaction term L_{ph-F} Eq. (4.66) in addition to the rules Eqs. (4.93), (4.94) and (4.95) we need to take care of the order of terms. Indeed if we "sandwich" the whole expression between $\bar{\psi}$ and ψ (here and below we perform calculations for right-moving impurities only, so we skip the + index in ψ_+) we obtain

$$L_{ph-F} \to \frac{\Gamma_0'}{2\pi} \frac{c}{m^*} \int dx\, "\bar{\psi}(x) \left[\left(\frac{\partial \chi_+(x,t)}{\partial x} \right) \left(\frac{\partial \chi_-(x,t)}{\partial x} \right) P^2 \right] \psi(x)". \tag{4.97}$$

In Eq. (4.97) we put the quotation marks since the terms in the square brackets can be permuted at the single impurity level to result in four different second-quantized expressions,

$$\left(\frac{\partial \chi_+(x,t)}{\partial x} \right) \left(\frac{\partial \chi_-(x,t)}{\partial x} \right) P^2,\ P^2 \left(\frac{\partial \chi_+(x,t)}{\partial x} \right) \left(\frac{\partial \chi_-(x,t)}{\partial x} \right),$$

$$P \left(\frac{\partial \chi_+(x,t)}{\partial x} \right) P \left(\frac{\partial \chi_-(x,t)}{\partial x} \right),\ \left(\frac{\partial \chi_+(x,t)}{\partial x} \right) P \left(\frac{\partial \chi_-(x,t)}{\partial x} \right) P. \tag{4.98}$$

We use the "democratic" symmetrized version

$$\bar{\psi}(x) \left(\overleftrightarrow{P} \right)^2 \psi(x) = -\frac{\hbar^2}{4} \left(\frac{\partial^2 \bar{\psi}(x)}{\partial x^2} \psi(x) - 2 \frac{\partial \bar{\psi}(x)}{\partial x} \frac{\partial \psi(x)}{\partial x} + \bar{\psi}(x) \frac{\partial^2 \psi(x)}{\partial x^2} \right). \tag{4.99}$$

Thus, finally, the second quantized action for (right- and left-moving) phonons and right-moving impurities is

$$\begin{aligned} S &= \int dxdt\ (\mathcal{L}_{ph} + \mathcal{L}_F + \mathcal{L}_{ph-F}) \\ &= \int dxdt\, \mathcal{L}_{ph} + \bar{\psi}(x)\left(i\hbar\frac{\partial}{\partial t} + ic\hbar\frac{\partial}{\partial x} + \frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial x^2}\right)\psi(x) + \\ &\quad + \frac{\Gamma_0'}{2\pi}\frac{c}{m^*}\left(\frac{\partial\chi_+(x,t)}{\partial x}\right)\left(\frac{\partial\chi_-(x,t)}{\partial x}\right)\bar{\psi}(x)\left(\overleftrightarrow{P}\right)^2\psi(x), \end{aligned} \tag{4.100}$$

to which methods of field theory (particularly, non-equilibrium field theory) may be applied. For the sake of brevity we have included only the right-moving impurities in Eq. (4.100). This is permissible, since right and left moving impurities do not interact directly, so they can be treated separately in the lowest orders of perturbation theory.

Having the field theory with two types of excitations: the phonons and the particle-hole pairs, we now wish to study dynamics of excitations in the system. This can be done by writing down the quantum kinetic equation for each type of excitation [91, 92].

4.8 Field theory: kinetic equations I

The quantum kinetic equation is a quantum analogue of the classical Boltzmann equations,

$$\left(\frac{\partial}{\partial t} + v_p\frac{\partial}{\partial x}\right)n(x,p;t) = I_{coll}[n], \tag{4.101}$$

where $n(r,p;t)$ is the Wigner transform of the density matrix in the position representation [92], see Eqs. (4.103) and (4.104), $v(p) = \partial E(p,n)/\partial p$ is the velocity at momentum p and the rest of the terms are grouped in the so-called collision integral $I_{coll}[n]$, in a direct analogy to the classical Boltzmann equation.

Because of the Heisenberg uncertainty principle, for quantum particles it is impossible to simultaneously specify the distribution function in both coordinate and momentum spaces. However, it can be show that starting with the one-particle density matrix $\rho(x_1, x_2; t)$ one can derive a function $n(x, p; t)$ that possesses certain properties of classical distribution functions. Indeed, let us take the one-particle density matrix $\rho(x_1, x_2; t)$ and instead of x_1 and x_2 make it a function of the "absolute" and the "relative" coordinates,

$$x = \frac{x_1 + x_2}{2}, \qquad x' = x_1 - x_2, \tag{4.102}$$

$$x_1 = x + \frac{x'}{2}, \qquad x_2 = x - \frac{x'}{2},$$

$$\rho(x_1, x_2; t) \to \rho(x, x'; t),$$

where the same letter ρ is used to denote both the function of x_1, x_2 and the function of x, x'. Change of variables, Eq. (4.102), is especially convenient when discussing spatially homogenous systems: one would expect the absolute coordinate x to be irrelevant and the one–particle density matrix to depend on x' only,

$$\rho(x_1, x_2; t) \to \rho(x'; t).$$

We can next define the Fourier transform of $\rho(x, x'; t)$ with respect to the relative coordinate x':

$$\frac{1}{N} n(x, p; t) = \int e^{-ipx'/\hbar} \rho\left(x + \frac{x'}{2}, x - \frac{x'}{2}; t\right) dx' \tag{4.103}$$

where N is the total number of particles. The inverse transform is

$$\rho\left(x_1, x_2; t\right) = \frac{1}{NL} \sum_p e^{ip(x_1-x_2)/\hbar} n\left(\frac{x_1+x_2}{2}, p; t\right) = \frac{1}{N} \int \frac{dp}{2\pi\hbar}\, e^{ip(x_1-x_2)/\hbar} n\left(\frac{x_1+x_2}{2}, p; t\right). \tag{4.104}$$

Equations (4.103) and (4.104) are also called Wigner transform and inverse Wigner transforms (in momentum space) [91].

Integrating $n(x, p; t)$ in x yields the distribution of particles in momentum space,

$$N_p = \int n(x, p; t)\, dx = N \int e^{-ip(x_1-x_2)/\hbar} \rho(x_1, x_2; t)\, dx_1 dx_2.$$

On the other hand, integrating $n(x, p; t)$ in p we get

$$N(x, t) = \int n(x, p; t)\, dp = N\rho(x, x; t),$$

the spatial distribution. Moreover, in the semiclassical limit $n(x, p; t)$ has the literal meaning of the distribution function [92], in that the average of any one-particle operator $O(x, -i\hbar\partial/\partial x)$ is given by

$$\langle O \rangle = \frac{1}{N} \int dx \frac{dp}{2\pi}\, O(x, p) n(x, p; t), \tag{4.105}$$

where $O(x, p)$ is the classical expression corresponding to $O(x, -i\hbar\partial/\partial x)$. The property Eq. (4.105) explains the reason for defining $n(x, p; t)$ in the first place: it is the quantum-mechanical analogue of the classical distribution function in phase space [92]. Again, it is only "analogue", since quantum mechanics prohibits a particle to simultaneously have definite coordinate and momentum values. In general, $n(x, p; t)$ is not even positive definite.

The quantum kinetic equation Eq. (4.101) is conveniently derived using the Keldysh formalism of non-equilibrium field theory [91, 92]. In the following Sections where all the calculations

are actually performed, we outline the main ideas behind Keldysh field theory in a more systematic manner (see Section 4.9 and [91] for detailed exposition). In what follows in this Section we rather state results without going into detailed derivations, not to lose focus.

Green's functions in the Keldysh theory are the two-by-two matrices

$$\hat{G}(\mathbf{x_1}, \mathbf{x_2}) = \begin{pmatrix} G^R(\mathbf{x_1}, \mathbf{x_2}) & G^K(\mathbf{x_1}, \mathbf{x_2}) \\ 0 & G^A(\mathbf{x_1}, \mathbf{x_2}) \end{pmatrix}, \tag{4.106}$$

with $G^{R/A}$ being ordinary retarder/advanced correlators and G^K is the Keldysh component of the Green's function. $\mathbf{x}$ stands for the pair (x, t). Eq. (4.106) is specific to fermions, for bosons (and in particular, phonons) the Keldysh propagator has the form

$$\hat{D}(\mathbf{x_1}, \mathbf{x_2}) = \begin{pmatrix} D^K(\mathbf{x_1}, \mathbf{x_2}) & D^R(\mathbf{x_1}, \mathbf{x_2}) \\ D^A(\mathbf{x_1}, \mathbf{x_2}) & 0 \end{pmatrix}. \tag{4.107}$$

Keldysh components of the propagator, G^K or D^K, are anti-Hermitean in the $(\mathbf{x_1}, \mathbf{x_2})$ space,

$$\left(G^K(\mathbf{x_1}, \mathbf{x_2})\right)^* = G^K(\mathbf{x_2}, \mathbf{x_1}), \tag{4.108}$$

and can be parameterized by a Hermitean function $F(\mathbf{x_1}, \mathbf{x_2})$:

$$G^K(\mathbf{x_1}, \mathbf{x_2}) = \int dx_3 dt_3 \left[G^R(\mathbf{x_1}, \mathbf{x_3}) F(\mathbf{x_3}, \mathbf{x_2}) - F(\mathbf{x_1}, \mathbf{x_3}) G^A(\mathbf{x_3}, \mathbf{x_2})\right]. \tag{4.109}$$

It turns out [91, 92] that a partial Wigner transform of the function $F(\mathbf{x_1}, \mathbf{x_2})$,

$$F(x, p; t) = \int \frac{d\omega}{2\pi} \left[\int dx' dt' \, e^{ipx'/\hbar - i\omega t'} F(x + \frac{x'}{2}, t + \frac{t'}{2}, x - \frac{x'}{2}, t - \frac{t'}{2})\right] \tag{4.110}$$

is related to $n(x,p;t)$ defined in Eq. (4.103). Namely, for fermions the relation is

$$F(x,p;t) = 1 - 2n_F(x,p;t) \tag{4.111}$$

and for bosons it is

$$f(x,q;t) = 1 + 2n_B(x,q;t). \tag{4.112}$$

To avoid confusion, here and elsewhere we use $F_\pm(x,p;t)$ for fermions (depletons) and $f_\pm(x,q;t)$ for bosons (phonons) with subscripts $\pm$ standing for right- and left- movers.

In an analogy to $T = 0$ and Matsubara formalisms, the matrix Green's functions Eqs. (4.106) and (4.107) obey Keldysh (matrix) version of the Dyson equation,

$$\hat{G} = \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}, \tag{4.113}$$

where "$\circ$" stands for matrix multiplication in Keldysh space as well as integration,

$$\left[\hat{\Sigma} \circ \hat{G}\right](\mathbf{x_1}, \mathbf{x_2}) = \int \mathbf{dx_3} \begin{pmatrix} \Sigma^R(\mathbf{x_1}, \mathbf{x_3}) & \Sigma^K(\mathbf{x_1}, \mathbf{x_3}) \\ 0 & \Sigma^A(\mathbf{x_1}, \mathbf{x_3}) \end{pmatrix} \begin{pmatrix} G^R(\mathbf{x_3}, \mathbf{x_2}) & G^K(\mathbf{x_3}, \mathbf{x_2}) \\ 0 & G^A(\mathbf{x_3}, \mathbf{x_2}) \end{pmatrix}. \tag{4.114}$$

The Keldysh component of Eq. (4.113) **is** the quantum kinetic equation for the function $F(\mathbf{x_1}, \mathbf{x_2})$ and taking the Wigner transforms in Eq. (4.110) we arrive at an equation analogous to Eq. (4.101),

$$\left(\frac{\partial}{\partial t} + v_p \frac{\partial}{\partial x}\right) F(x,p;t) \approx I_{coll}[F], \tag{4.115}$$

with the right hand side (the collision integral) given by

$$I_{coll}[F] = i\Sigma^K(x,p;t) + 2F(x,p;t)\mathrm{Im}\Sigma^R(x,p;t), \tag{4.116}$$

with $\Sigma^R(x,p;t)$, $\Sigma^K(r,p;t)$ being the Wigner transforms of the retarded and Keldysh components of self-energy, taken on the mass shell,

$$\Sigma(x,p;t,\omega) \to \Sigma(x,p;t,E(p)/\hbar). \tag{4.117}$$

For example, for non-interacting particles have $\hat{\Sigma}=0$ and the collision integral vanishes identically. This means that "distributions" $F(x,p;t)$ for each fixed p evolve independently of each other and the number of particles with given momentum is conserved.

In Eqs. (4.115) and (4.116) we used notation appropriate to fermionic fields, G and Σ, however everything also holds for bosonic Keldysh Green's functions $\hat{D}_\pm$ and phononic polarization operators $\hat{\Pi}_\pm$. Keldysh components of the Dyson equations for $\hat{D}_\pm$ result in kinetic equations similar to Eq. (4.115):

$$\left(\frac{\partial}{\partial t} \pm c\frac{\partial}{\partial x}\right) f_\pm(x,q;t) \approx i\Pi^K_\pm(x,q;t) + 2f_\pm(x,q;t)\mathrm{Im}\Pi^R_\pm(x,q;t). \tag{4.118}$$

Similar to the case of non-linear phonons of Sec. 4.3, important physics is contained in the self-energy functions, either depletonic or phononic, which we also call the polarization operator. This time, however, all the bare Green's functions are not linear and the self-energies are no longer divergent. In the following Sections we obtain the self-energies for both depletons and phonons and see how including depletons modifies the dynamics.

4.9 Field theory: Keldysh formalism

A short summary of the non–equilibrium field theory is in place here. Keldysh field theory is also known by the name "field theory on the closed time contour". This means that instead of conventional action Eq. (4.100), where the time integral of the Lagrangian goes from $-\infty$ to

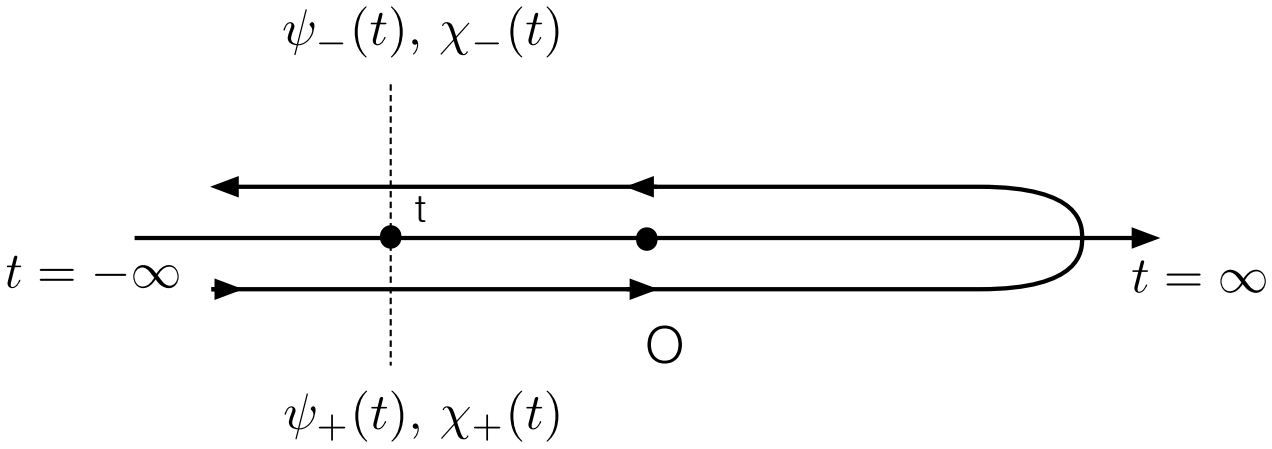


Figure 4.6: In non-equilibrium field theory time integration first goes from $t = -\infty$ to $t = \infty$, and then back. Values of the fields on the two branches are independent, which effectively doubles the number of fields.

$+\infty$, we extend the time integration to the line that also goes backwards in time, from $+\infty$ to $-\infty$, thus closing the contour (see Fig. 4.6):

$$S = \oint dxdt\ \left(\mathcal{L}_F + \mathcal{L}_{ph} + \mathcal{L}_{ph-F}\right). \tag{4.119}$$

Here $\oint$ stands for integration from $-\infty$ to ∞ and then backwards,

$$\oint dt = \int_{-\infty}^{\infty} dt + \int_{\infty}^{-\infty} dt = \int_{-\infty}^{\infty} dt - \int_{-\infty}^{\infty} dt. \tag{4.120}$$

This effectively doubles the number of fields, or rather gives every field an additional two–valued index, which distinguishes between the values taken at the same point in time on the forward and backward going contours (Fig. 4.6). It is convenient to perform the so–called Keldysh rotation on the pairs of $(+,-)$ fields. For real-valued bosons we rotate from $(+,-)$ to (cl,q)[1]

[1] cl stands for "classical" and q stands for "quantum", see [91] for the reasoning behind this choice of labels, and also for the reasoning why fermionic fields are labelled with $(1,2)$ indices and not (cl,q).

space,

$$\begin{pmatrix} \chi^{cl} \\ \chi^{q} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \chi^{+} \\ \chi^{-} \end{pmatrix}, \qquad \begin{pmatrix} \chi^{+} \\ \chi^{-} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \chi^{cl} \\ \chi^{q} \end{pmatrix} \tag{4.121}$$

and for fermions we go from $(+,-)$ to $(1,2)$ space,

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \psi_{+} \\ \psi_{-} \end{pmatrix}, \qquad \begin{pmatrix} \psi_{+} \\ \psi_{-} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \bar{\psi}_{+} \\ \bar{\psi}_{-} \end{pmatrix}, \qquad \begin{pmatrix} \bar{\psi}_{+} \\ \bar{\psi}_{-} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix}. \tag{4.122}$$

Now that we have kept only right-handed impurities, one should not confuse the fermionic field variables on the forward and backward contours with the right- and left-moving impurity fields. Same goes for phononic fields χ, for which Keldysh indices are written in the superscript and chiral indices go in the subscript.

The reasoning behind Keldysh rotations is that the non-equilibrium Green's functions take more convenient form in $(1,2)$, (cl,q) spaces than in $(+,-)$ space. Conventional zero–temperature or Matsubara Green's functions are averages of two field operators, so when we extend the notion of Green's functions to the non–equilibrium theory, they also acquire two additional indices, one for each field variable, thus making Green's functions matrices in the Keldysh space [91, 93],

$$\hat{G} = G_{ab}(\mathbf{x_1}, \mathbf{x_2}) = \begin{pmatrix} G^R(\mathbf{x_1}, \mathbf{x_2}) & G^K(\mathbf{x_1}, \mathbf{x_2}) \\ 0 & G^A(\mathbf{x_1}, \mathbf{x_2}) \end{pmatrix},$$

$$\hat{D} = D^{\alpha\beta}(\mathbf{x_1}, \mathbf{x_2}) = \begin{pmatrix} D^K(\mathbf{x_1}, \mathbf{x_2}) & D^R(\mathbf{x_1}, \mathbf{x_2}) \\ D^A(\mathbf{x_1}, \mathbf{x_2}) & 0 \end{pmatrix}, \tag{4.123}$$

where for fermionic Green's functions $\hat{G}$ indices (a, b) take the values from the set $(1, 2)$ and for the phononic Green's functions $\hat{D}$ indices (α, β) take the values from the set (cl, q) and boldface $\mathbf{x}$ stand for the pairs (x, t), compare with Section 4.8. Due to Keldysh rotation performed, one of the components of the matrices Eq. (4.123) is identically zero and two of the remaining components are just conventional retarded and advanced Green's functions and are related by complex conjugation (see e.g. Eq. (4.19)). Self-energy and polarization operators also become two-by-two matrices

$$\hat{\Sigma} = \Sigma_{ab}(\mathbf{x_1}, \mathbf{x_2}) = \begin{pmatrix} \Sigma^R(\mathbf{x_1}, \mathbf{x_2}) & \Sigma^K(\mathbf{x_1}, \mathbf{x_2}) \\ 0 & \Sigma^A(\mathbf{x_1}, \mathbf{x_2}) \end{pmatrix},$$

$$\hat{\Pi} = \Pi^{\alpha\beta} = \begin{pmatrix} 0 & \Pi^A(\mathbf{x_1}, \mathbf{x_2}) \\ \Pi^R(\mathbf{x_1}, \mathbf{x_2}) & \Pi^K(\mathbf{x_1}, \mathbf{x_2}) \end{pmatrix}. \tag{4.124}$$

In order to actually calculate the self-energy for the collision integral Eq. (4.116) in the lowest order in interaction vertices L_{ph-F}, we use the perturbative expansion of the Dyson equation

$$\hat{G} = \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G} = \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 + \dots$$

and we remind that "$\circ$" stands for both matrix multiplication in the Keldysh space and in real space, see Eq. (4.114).

In homogeneous environment we expect the system to be translationally invariant, and hence expect the Green's functions and self-energies to depend only on the differences of spatial and

temporal coordinates,

$$G(x_1, t_1; x_2, t_2) \to G(x_1 - x_2, t_1 - t_2). \tag{4.125}$$

Wigner transforms then become ordinary Fourier transforms,

$$G(x,t) = \int \frac{dkd\epsilon}{(2\pi)^2}\, e^{ikx - i\epsilon t} G(k,\epsilon). \tag{4.126}$$

For example, if

$$G^R(x,t) = -i \left\langle \psi_1(x,t)\bar{\psi}_1(0,0) \right\rangle, \tag{4.127}$$

then

$$(2\pi)^2\, G^R(k,\epsilon)\delta(k-k')\delta(\epsilon-\epsilon') = -i \left\langle \psi_1(k,\epsilon)\bar{\psi}_1(k',\epsilon') \right\rangle. \tag{4.128}$$

Notice that for fermions k and ϵ stand for the wavenumber and frequency, see Eq. (4.126), not wavenumber and energy. In the non-interacting case, $\mathcal{L}_{ph-F} = 0$, retarded and advanced fermionic Green's functions $\hat{G}_0$ are given by

$$G^R_{0,+}(k,\epsilon) = \left(G^A_{0,+}(k,\epsilon)\right)^\dagger = \frac{1}{\epsilon - \epsilon_+(k)/\hbar + i\cdot 0} \tag{4.129}$$

for the right (+) movers and

$$G^R_{0,-}(k,\epsilon) = \left(G^A_{0,-}(k,\epsilon)\right)^\dagger = \frac{1}{\epsilon - \epsilon_-(k)/\hbar + i\cdot 0} \tag{4.130}$$

for the left (−) movers, where $\epsilon(k) = E(\hbar k)/\hbar$. For example, for the right movers

$$\epsilon_+(k) = ck + \frac{\hbar k^2}{2m*}. \tag{4.131}$$

Eqs. (4.129) and (4.130) can be verified in a way analogous to deriving Eq. (4.21), by writing

the non-interacting fermionic action in terms of Fourier components and performing the Grassmanian Gaussian path integrals. The Keldysh components are parameterized by "distribution functions" $F_+(\mathbf{x_1}, \mathbf{x_2})$, $F_-(\mathbf{x_1}, \mathbf{x_2})$, see Eqs. (4.109), (4.111) and (4.112) and discussion of Section 4.8. If the distributions are homogeneous in space and time Wigner transform is equivalent to Fourier transform, and the "$\circ$"-product becomes the ordinary multiplication:

$$G^K_{0,+}(k,\epsilon) = F_+(k,\epsilon)\left(G^R_{0,+}(k,\epsilon) - G^A_{0,+}(k,\epsilon)\right), \tag{4.132}$$

$$G^K_{0,-}(k,\epsilon) = F_-(k,\epsilon)\left(G^R_{0,-}(k,\epsilon) - G^A_{0,-}(k,\epsilon)\right). \tag{4.133}$$

Non-interacting phononic Green's functions $\hat{D}_0$ are familiar from Section 4.2,

$$D^R_{0,+}(q,\omega) = \left(D^A_{0,+}(q,\omega)\right)^\dagger = \frac{\pi}{q(\omega - cq + i\cdot 0)} \tag{4.134}$$

and

$$D^R_{0,-}(q,\omega) = \left(D^A_{0,-}(q,\omega)\right)^\dagger = \frac{\pi}{-q(\omega + cq + i\cdot 0)} \tag{4.135}$$

for the right $(+)$ and left $(-)$ moving phonons. Similarly to Eqs. (4.132) and (4.133), the Keldysh components of phononic propagators are

$$D^K_{0,+} = f_+(q,\omega)\left(D^R_{0,+}(q,\omega) - D^A_{0,+}(q,\omega)\right) \tag{4.136}$$

and

$$D^K_{0,-} = f_-(q,\omega)\left(D^R_{0,-}(q,\omega) - D^A_{0,-}(q,\omega)\right). \tag{4.137}$$

We finally derive the Keldysh form of the interaction term $\mathcal{L}_{ph-F}$, Eq. (4.100), by extending

the time integral to the closed contour:

$$\int dt \left[\int dx \frac{1}{2\pi} \frac{c}{m^*} \Gamma_0' \frac{\partial \chi_+}{\partial x} \frac{\partial \chi_-}{\partial x} \left(\bar{\psi} \left(\overleftrightarrow{P} \right)^2 \psi \right) \right] \to \oint dt \left[\int dx \frac{1}{2\pi} \frac{c}{m^*} \Gamma_0' \frac{\partial \chi_+}{\partial x} \frac{\partial \chi_-}{\partial x} \left(\bar{\psi} \left(\overleftrightarrow{P} \right)^2 \psi \right) \right]. \tag{4.138}$$

When we expand the time contour integral Eq. (4.138) in line with Eq. (4.120), we double the number of fields: χ becomes χ^+ on the forward–going slice and χ^- on the backward–going slice. ψ goes into ψ_+ and ψ_-, respectively. For the sake of brevity we neglect all the prefactors and derivative operators in Eq. (4.138) and reduce it to

$$\int dt \left[\int dx \frac{1}{2\pi} \frac{c}{m^*} \Gamma_0' \frac{\partial \chi_+}{\partial x} \frac{\partial \chi_-}{\partial x} \left(\bar{\psi} \left(\overleftrightarrow{P} \right)^2 \psi \right) \right] \to \int dx dt \, \chi_+ \chi_- \bar{\psi} \psi. \tag{4.139}$$

Then the Keldysh form of Eq. (4.139) is

$$\int dx dt \, \chi_+ \chi_- \bar{\psi} \psi \to \oint dx dt \, \chi_+ \chi_- \bar{\psi} \psi = \int dx dt \, \left(\chi_+^+ \chi_-^+ \bar{\psi}^+ \psi^+ - \chi_+^- \chi_-^- \bar{\psi}^- \psi^- \right). \tag{4.140}$$

The interaction term splits into the following components:

$$\begin{aligned} \oint dx dt \, \chi_+ \chi_- \bar{\psi} \psi \;\; = \;\; & \int dx dt \, \left[\chi_+^{cl} \chi_-^{cl} \bar{\psi}_1 \psi_1 + \chi_+^{cl} \chi_-^{cl} \bar{\psi}_2 \psi_2 + \chi_+^q \chi_-^q \bar{\psi}_1 \psi_1 + \chi_+^q \chi_-^q \bar{\psi}_2 \psi_2 + \right. \\ & \left. + \chi_+^{cl} \chi_-^q \bar{\psi}_1 \psi_2 + \chi_+^{cl} \chi_-^q \bar{\psi}_2 \psi_1 + \chi_+^q \chi_-^{cl} \bar{\psi}_1 \psi_2 + \chi_+^q \chi_-^{cl} \bar{\psi}_2 \psi_1 \right] \end{aligned} \tag{4.141}$$

The sum of 8 terms with different combinations of indices (cl, q) and $(1, 2)$ can be represented as a tensor contraction:

$$\oint dx dt \, \chi_+ \chi_- \bar{\psi} \psi = \int dx dt \, \gamma_{\alpha,\beta}^{ab} \chi_+^\alpha \chi_-^\beta \bar{\psi}_a \psi_b, \tag{4.142}$$

$$\frac{\hbar^2}{8\pi}\frac{c\Gamma_0'}{m^*}(-qq')\left(2k+q+q'\right)^2\gamma^{ab}_{\alpha,\beta}$$

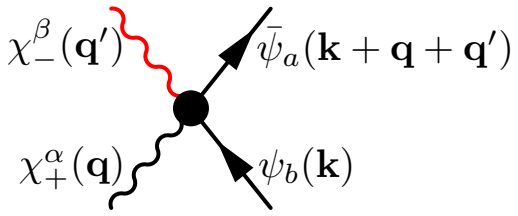


Figure 4.7: The vertex corresponding to the phonon-fermion interaction Eq. (4.145). Boldface $\mathbf{q}$, $\mathbf{k}$ stand for the phonon and fermion wavenumber–frequency pairs $\mathbf{q}=(q,\omega)$, $\mathbf{k}=(k,\epsilon)$. Momenta and energies of phonons and ψ should be considered incoming, and momentum/energy of $\bar{\psi}$ should be considered outgoing. With such convention the momentum and energy at the vertex are conserved.

where

$$\hat{\gamma}_{cl,cl}=\hat{\gamma}_{q,q}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \hat{\gamma}_{q,cl}=\hat{\gamma}_{cl,q}=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{4.143}$$

Returning to the full notation, the interaction term(s) in Keldysh formalism is

$$S_{ph-F}=\int dxdt\left[\frac{1}{2\pi}\frac{c}{m^*}\Gamma_0'\gamma^{ab}_{\alpha,\beta}\frac{\partial\chi^{\alpha}_{+}}{\partial x}\frac{\partial\chi^{\beta}_{-}}{\partial x}\bar{\psi}_a\left(\overleftrightarrow{P}\right)^2\psi_b\right], \tag{4.144}$$

where summation over repeated greek and latin indices is assumed. The interaction term(s) may be written in terms of the Fourier components of the fields,

$$\begin{aligned}S_{ph-F} &= \int\frac{dqd\omega}{(2\pi)^2}\frac{dq'd\omega'}{(2\pi)^2}\frac{dkd\epsilon}{(2\pi)^2}\frac{\hbar^2}{8\pi}\frac{c\Gamma_0'}{m^*}\gamma^{ab}_{\alpha,\beta}(-qq')\left(2k+q+q'\right)^2\times\\ &\times\left[\bar{\psi}_a(k+q+q',\epsilon+\omega+\omega')\psi_b(k,\epsilon)\chi^{\alpha}_{+}(q,\omega)\chi^{\beta}_{-}(q',\omega')\right],\end{aligned} \tag{4.145}$$

where we have used the continuous version for convenience, to avoid working with sums and rather use integrals. Eq. (4.145) is bulky but it invites intuitive diagrammatic interpretation of a vertex that conserves momentum and energy, Fig. (4.7).

4.10 Field theory: self-energies

The important physics is still contained in the fermionic and phononic self-energies. However, unlike in the case of interacting phonons, Section 4.2, the perturbation theory is not resonant since the interaction vertex no longer describes collisions between phonons alone, but rather collisions between linear phonons and quadratic impurities.

The summary of Keldysh theory outlined in Sec. 4.9 highlights the main differences between the conventional zero-temperature or Matsubara formalism and Keldysh field theory: a) the Green's functions become two-by-two matrices and b) instead of one interaction term in the action we get several (eight in the case considered, Eq. (4.141)), corresponding to a number of different combinations of Keldysh indices. Consequently, in the perturbation expansion of the interaction term

$$\exp\left(\frac{i}{\hbar}S_{ph-F}\right) = 1 + \frac{i}{\hbar}S_{ph-F} - \frac{1}{2\hbar^2}\left(S_{ph-F}\right)^2 + ... \tag{4.146}$$

up to the first, second, etc order, the terms S_{ph-F}, $\left(S_{ph-F}\right)^2$, ... contain 8, 64, ... combinations of the product terms which all need to be summed. Fortunately, using the $\hat{\gamma}$-matrix notation Eqs. (4.143), at least the notation is no more complicated than that of conventional perturbation theory. In addition, some of these combinations might evaluate to 0 under averaging, for example in Keldysh theory all the terms represented by disconnected diagrams are identically zero [91]. The advantage of using Keldysh formalism, however, is that the results are not limited to the case of thermal equilibrium and quantum kinetic equation is obtained automatically, as one component of two-by-two Keldysh-Dyson equation, see Section 4.8.

Let us now apply the perturbation theory to evaluate the self-energies and polarization operators of the right-moving phonons and impurities, relevant to Eqs. (4.115), (4.116) and

(4.118). By definition, the full fermionic Green's function is

$$G_{ab}(x_1, x_2) = -i \int \mathcal{D}[\psi]\ \psi_a(x_1)\bar{\psi}_b(x_2) \exp\left(\frac{i}{\hbar}\left(S_0 + S_{ph-F}\right)\right), \tag{4.147}$$

where $\mathcal{D}[\psi]$ is the measure of the Grassman path integral over the Keldysh fields ψ_1, ψ_2 and their conjugates,

$$\mathcal{D}[\psi] = \mathcal{D}\bar{\psi}_1 \mathcal{D}\psi_1 \mathcal{D}\bar{\psi}_2 \mathcal{D}\psi_2. \tag{4.148}$$

Expanding Eq. (4.147) in the series Eq. (4.146),

$$\begin{aligned} G_{ab}(x_1, t_1; x_2, t_2) \;=\; & G_{0,ab}(x_1, t_1; x_2, t_2) - \frac{1}{\hbar}\int \mathcal{D}[\psi]\ \psi_a(x_1, t_1)\bar{\psi}_b(x_2, t_2)\left(S_{ph-F}\right) \exp\left(\frac{i}{\hbar} S_0\right) + \\ & + \frac{i}{2\hbar^2}\int \mathcal{D}[\psi]\ \psi_a(x_1, t_1)\bar{\psi}_b(x_2, t_2)\left(S_{ph-F}\right)^2 \exp\left(\frac{i}{\hbar} S_0\right), \end{aligned} \tag{4.149}$$

we compare it to the Dyson equation to evaluate the impurity self-energy $\hat{\Sigma}$:

$$\hat{G} = \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G} = \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 + \dots . \tag{4.150}$$

One can discard the first order term in S_{ph-F} in Eq. (4.149), since it involves the averages

$$\langle \chi_+ \rangle_0 , \qquad \langle \chi_- \rangle_0 , \tag{4.151}$$

and therefore evaluates to zero. Then, comparing Eqs. (4.150) and (4.149), and remembering that in Keldysh technique the disconnected diagrams of the perturbation series evaluate to

zero, we obtain the expression for the self-energy in the lowest order of perturbation theory,

$$\begin{aligned}\left(\hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0\right)_{ab}(x_1,t_1;x_2,t_2) &= \sum_{c,d=1}^{2}\int dxdtdx'dt'\, G_{0,ac}(x_1,t_1;x,t)\Sigma_{cd}(x,t;x',t')G_{0,db}(x',t';x_2,t_2) \\ &= \frac{i}{2\hbar^2}\int \mathcal{D}\left[\psi\right]\, \psi_a(x_1,t_1)\bar{\psi}_b(x_2,t_2)\left(S_{ph-F}\right)^2 \exp\left(\frac{i}{\hbar}S_0\right). \qquad (4.152)\end{aligned}$$

Due to presence of a relatively complicated second derivative term $\bar{\psi}_a\left(\overleftrightarrow{P}\right)^2\psi_b$, it is easier to perform the calculations in the Fourier space, where the derivation becomes multiplication. Fourier transforming Eq. (4.152) from $x_1 - x_2$ and $t_1 - t_2$ to k and ϵ we obtain

$$\sum_{c,d=1}^{2} G_{0,ac}(k,\epsilon)\Sigma_{cd}(k,\epsilon)G_{0,db}(k,\epsilon) = \frac{i}{2\hbar^2}\int\frac{dk'd\epsilon'}{(2\pi)^2}\int \mathcal{D}\left[\psi\right]\, \psi_a(k,\epsilon)\bar{\psi}_b(k',\epsilon')\left(S_{ph-F}\right)^2\exp\left(\frac{i}{\hbar}S_0\right), \qquad (4.153)$$

where we need to use the Fourier-space version of S_{ph-F}, Eq. (4.145). Using the shorthand notation

$$\int \mathcal{D}\left[\psi\right]\, \ldots \exp\left(\frac{i}{\hbar}S_0\right) \leftrightarrow \left\langle \ldots \right\rangle_0$$

we expand the right hand term of Eq. (4.153):

$$\begin{aligned}\frac{i}{2\hbar^2}\left(\frac{\hbar^2}{8\pi}\frac{c\Gamma_0'}{m^*}\right)^2\gamma^{mn}_{\alpha,\beta}\gamma^{pr}_{\gamma,\delta} &= \int\frac{dq_1d\omega_1}{(2\pi)^2}\frac{dq_1'd\omega_1'}{(2\pi)^2}\frac{dk_1d\epsilon_1}{(2\pi)^2}\frac{dq_2d\omega_2}{(2\pi)^2}\frac{dq_2'd\omega_2'}{(2\pi)^2}\frac{dk_2d\epsilon_2}{(2\pi)^2}\ \times \\ &\times(-q_1q_1')(-q_2q_2')\left(2k_1+q_1+q_1'\right)^2\left(2k_2+q_2+q_2'\right)^2\times \\ &\times\Big\langle\chi_+^\alpha(q_1,\omega_1)\chi_+^\gamma(q_2,\omega_2)\chi_-^\beta(q_1',\omega_1')\chi_-^\delta(q_2',\omega_2')\psi_a(k,\epsilon)\bar{\psi}_b(k,\epsilon)\ldots \\ &\ldots\bar{\psi}_m(k_1+q_1+q_1',\epsilon_1+\omega_1+\omega_1')\psi_n(k_1,\epsilon_1)\ldots \\ &\ldots\bar{\psi}_p(k_2+q_2+q_2',\epsilon_2+\omega_2+\omega_2')\psi_r(k_2,\epsilon_2)\Big\rangle_0. \qquad (4.154)\end{aligned}$$

Since the averaging is done over the non-interacting action ($\langle\ldots\rangle_0$) the second line of Eq. (4.154)

separates into the product of averages:

$$\left\langle \chi_+^{\alpha}(q_1,\omega_1)\chi_+^{\gamma}(q_2,\omega_2)\right\rangle_0 = i(2\pi)^2 D_{0,+}^{\alpha\gamma}(q_1,\omega_1)\delta(q_1+q_2)\delta(\omega_1+\omega_2), \tag{4.155}$$

$$\left\langle \chi_-^{\beta}(q_1',\omega_1')\chi_-^{\delta}(q_2',\omega_2')\right\rangle_0 = i(2\pi)^2 D_{0,-}^{\beta\delta}(q_1',\omega_1')\delta(q_1'+q_2')\delta(\omega_1'+\omega_2'), \tag{4.156}$$

and

$$\left\langle \psi_a(k,\epsilon)\bar{\psi}_b(k,\epsilon)\bar{\psi}_m(k_1+q_1+q_1',\epsilon_1+\omega_1+\omega_1')\psi_n(k_1,\epsilon_1)\bar{\psi}_p(k_2+q_2+q_2',\epsilon_2+\omega_2+\omega_2')\psi_r(k_2,\epsilon_2)\right\rangle_0, \tag{4.157}$$

which can be evaluated using Wick's theorem for fermionic fields. Taking into account the minus sign due to anti–commutation of Grassman variables and combinatorial factor of 2 for the number of diagrams, we obtain

$$\begin{aligned}\Sigma_{ab}(k,\epsilon) &= -\left(\frac{\hbar c\Gamma_0'}{8\pi m^*}\right)^2 \hat{\gamma}_{\alpha,\beta}^{ac}\hat{\gamma}_{\gamma,\delta}^{db}\int \frac{dqd\omega}{(2\pi)^2}\frac{dq'd\omega'}{(2\pi)^2}(2k-q-q')^4(qq')^2 \times \\ &\quad \times G_{0,cd}(k-q-q',\epsilon-\omega-\omega')D_{0,+}^{\alpha\gamma}(q,\omega)D_{0,-}^{\beta\delta}(q',\omega'),\end{aligned} \tag{4.158}$$

where we took into account the energy-momentum conservation mentioned after Eq. (4.145) and properties of the Green's functions in the Fourier space Eq. (4.128) to at once integrate out four out of six internal momenta-energies. Then, according to Eq. (4.158), the retarded component of the self-energy, $\Sigma^R(k,\epsilon) = \Sigma_{11}(k,\epsilon)$, is given by

$$\begin{aligned}\Sigma^R(k,\epsilon) &= -\left(\frac{\hbar c\Gamma_0'}{8\pi m^*}\right)^2 \hat{\gamma}_{\alpha,\beta}^{1c}\hat{\gamma}_{\gamma,\delta}^{d1}\int \frac{dqd\omega}{(2\pi)^2}\frac{dq'd\omega'}{(2\pi)^2}(2k-q-q')^4(qq')^2 \times \\ &\quad \times G_{0,cd}(k-q-q',\epsilon-\omega-\omega')D_{0,+}^{\alpha\gamma}(q,\omega)D_{0,-}^{\beta\delta}(q',\omega').\end{aligned} \tag{4.159}$$

We can write out explicitly all the non–zero term present in Eq. (4.159) by observing, that G_{21},

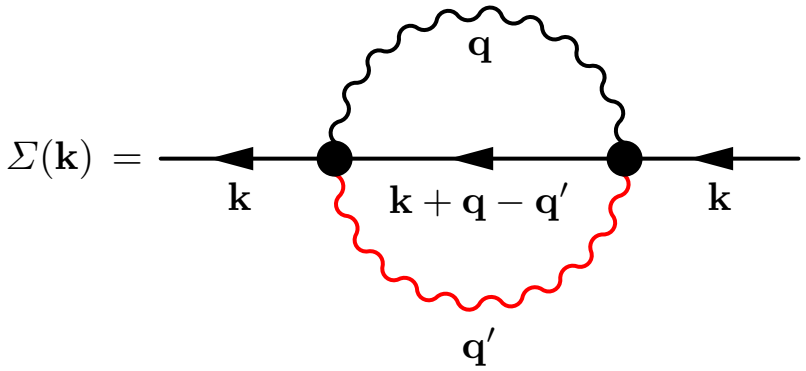


Figure 4.8: Feynman diagram corresponding to the fermionic self-energy in the lowest order of perturbation theory. Wiggly lines represent the bare phononic propagators, $D_{0,+}(\mathbf{q})$ (black) and $D_{0,-}(\mathbf{q}')$ (red). Solid lines represent the bare fermionic Green's functions G_0.

D^{qq} and several entries of matrices $\hat{\gamma}$ (Eqs. (4.143)) are identically equal to zero. This leaves only the following combinations under the integral sign (arguments of the Green's functions, as well as subscripts "0", have been omitted):

$$G^R D_+^K D_-^K + G^K D_+^K D_-^R + G^K D_+^R D_-^K + G^R D_+^R D_-^R + G^A D_+^A D_-^R + G^A D_+^R D_-^A + G^R D_+^A D_-^A. \quad (4.160)$$

Similarly, the Keldysh component $\Sigma^K(k,\epsilon) = \Sigma_{12}(k,\epsilon)$ is given by

$$\begin{aligned}\Sigma^K(k,\epsilon) &= -\left(\frac{\hbar c \Gamma_0'}{8\pi m^*}\right)^2 \int \frac{dq d\omega}{(2\pi)^2}\frac{dq' d\omega'}{(2\pi)^2}(2k-q-q')^4 (qq')^2 \times \\ &\times \left[G^K D_+^K D_-^K + G^R D_+^K D_-^R + G^R D_+^R D_-^K + G^A D_+^K D_-^A + \right. \\ &\left. + G^A D_+^A D_-^K + G^K D_+^R D_-^R + G^K D_+^A D_-^A\right]. \end{aligned} \quad (4.161)$$

Instead of doing the perturbation expansion explicitly, one could use diagrammatics, Fig. 4.8, which only differs from equilibrium diagrammatics by that all Green's functions now acquire Keldysh indices and all vertices contain a $\hat{\gamma}$-matrix each (compare to Fig. 4.7), so allowed

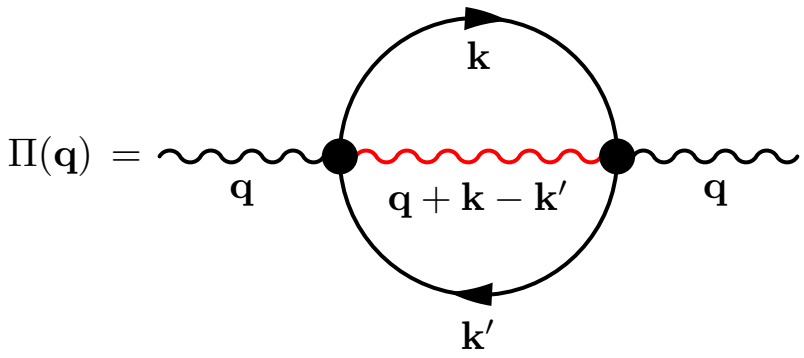


Figure 4.9: Feynman diagram corresponding to the phononic polarization operator in the lowest order of perturbation theory. Wiggly lines represent the bare phononic propagators, $D_{0,+}(\mathbf{q})$ (black) and $D_{0,-}(\mathbf{q}+\mathbf{k}-\mathbf{k}')$ (red). Solid lines represent the bare fermionic Green's functions G_0.

combinations of Green's functions need to be found and summed over, leading to expressions Eqs. (4.160), (4.161) and similar.

The phonon polarization operators can be calculated in a similar manner, by using the phononic Dyson equation

$$\hat{D}_{\pm} = \hat{D}_{0,\pm} + \hat{D}_{0,\pm} \circ \hat{\Pi}_{\pm} \circ \hat{D}_{0,\pm} + \dots \tag{4.162}$$

and comparing it to the perturbation series expansion $\left\langle \chi_{+}^{\alpha}(q,\omega)\chi_{+}^{\beta}(q',\omega') \right\rangle$.

Alternatively, $\hat{\Pi}$ can be evaluated with the help of diagrammatics, Fig. 4.9, keeping summation over the indices of $\hat{\gamma}$-matrices in mind, to obtain the expression very similar to Eq. (4.158),

$$\begin{aligned}\Pi_{\alpha\gamma}(q,\omega) &= \left(\frac{\hbar c \Gamma_0'}{8\pi m^*}\right)^2 \hat{\gamma}^{ab}_{\alpha,\beta}\hat{\gamma}^{cd}_{\gamma,\delta} \int \frac{dk d\epsilon dq' d\omega'}{(2\pi)^4} (qq')^2 (2k-q+q')^2 \times \\ &\times D^{\beta\delta}_{0,-}(q',\omega') G_{0,bc}(k,\epsilon) G_{0,da}(k-q+q', \epsilon-\omega+\omega').\end{aligned} \tag{4.163}$$

For example, the retarded component, $\Pi^R = \Pi_{q,cl}$ is given by the following combination:

$$\begin{aligned}\Pi^R(q,\omega) &= \left(\frac{\hbar c\Gamma_0'}{8\pi m^*}\right)^2 \int \frac{dkd\epsilon dq'd\omega'}{(2\pi)^4}(qq')^2(2k-q+q')^2 \times \\ &\times \left[D_-^R G^R G^A + D_-^R G^K G^K + D_-^R G^A G^R + D_-^A G^R G^R + \right. \\ &\left. + D_-^A G^A G^A + D_-^K G^K G^R + D_-^K G^A G^K\right]. \end{aligned} \tag{4.164}$$

Similarly, the expression for $\Pi_{q,q} = \Pi^K$ can be found:

$$\begin{aligned}\Pi^K(q,\omega) &= \left(\frac{\hbar c\Gamma_0'}{8\pi m^*}\right)^2 \int \frac{dkd\epsilon dq'd\omega'}{(2\pi)^4}(qq')^2(2k-q+q')^2 \times \\ &\times \left[D_-^R G^R G^K + D_-^R G^K G^A + D_-^A G^K G^R + D_-^A G^A G^K + \right. \\ &\left. + D_-^K G^R G^R + D_-^R G^A G^A + D_-^K G^K G^K\right]. \end{aligned} \tag{4.165}$$

The expressions Eqs. (4.160), (4.161), (4.164) and (4.165) can be evaluated by properly grouping the terms. Let us demonstrate this for Σ^K, Eq. (4.161), where the expression under the integral sign is

$$G^K D_+^K D_-^K + G^R D_+^K D_-^R + G^R D_+^R D_-^K + G^A D_+^K D_-^A + G^A D_+^A D_-^K + G^K D_+^R D_-^R + G^K D_+^A D_-^A, \tag{4.166}$$

and we remind that the arguments of the Green's functions are:

$$D_+ \to D_{0,+}(q,\omega), \tag{4.167}$$

$$D_- \to D_{0,-}(q',\omega'), \tag{4.168}$$

$$G \to G_0(k-q-q', \epsilon-\omega-\omega'). \tag{4.169}$$

The first term in Eq. (4.166) is just a product of three delta–functions and three distribution

functions, compare to Eqs. (4.129) and (4.132):

$$\begin{aligned} G^K D_+^K D_-^K &= F(k-q-q')(-2\pi i)\delta(\epsilon-\omega-\omega'-\epsilon_+(k-q-q'))\times \\ &\quad \times f_+(q)\left(-\frac{2\pi^2}{q}i\right)\delta(\omega-cq)f_+(q')\left(\frac{2\pi^2}{q'}i\right)\delta(\omega'+cq'). \end{aligned} \tag{4.170}$$

The remaining six terms can be grouped as follows:

$$D_+^K\left(G^R D_-^R + G^A D_-^A\right) + D_-^K\left(G^R D_+^R + G^A D_+^A\right) + G^K\left(D_+^R D_-^R + D_+^A D_-^A\right). \tag{4.171}$$

Consider, for example,

$$G^R D_-^R + G^A D_-^A = \left(G^R - G^A\right)\left(D_-^R - D_-^A\right) + G^R D_-^A + G^A D_-^R. \tag{4.172}$$

In Eq. (4.172) the products $G^R D_-^A$ and $G^A D_-^R$ have different causal structure and therefore evaluate to zero (in direct space these combinations would be analogous to having the combination[2] "$\theta(t-t')\theta(t'-t)=0$"), therefore the following relation holds:

$$G^R D_-^R + G^A D_-^A = \left(G^R - G^A\right)\left(D_-^R - D_-^A\right) = (-2\pi i)\delta(\epsilon-\omega-\omega'-\epsilon_+(k-q-q'))\left(\frac{2\pi^2}{q'}i\right)\delta(\omega'+cq'). \tag{4.173}$$

Similiar reasoning can be applied to the two remaining pairs and the following result is obtained for Eq. (4.166):

$$\begin{aligned} &\left(-\frac{8\pi^5}{qq'}i\right)\delta(\epsilon-\omega-\omega'-\epsilon_+(k-q-q'))\delta(\omega-cq)\delta(\omega'+cq')\times \\ &\times\left[F(k-q-q')f_+(q)f_-(q') + F(k-q-q') + f_+(q) + f_-(q')\right]. \end{aligned} \tag{4.174}$$

[2] $\theta(t)$ is the step function, defined as $\theta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$.

Two of the three $\delta-$functions present in Eq. (4.174) allow to integrate out ω and ω' in Eq. (4.158), resulting in

$$\Sigma^K(k,\epsilon) = i\frac{\pi^3}{8}\left(\frac{\hbar c\Gamma_0'}{m^*}\right)^2 \int \frac{dqdq'}{(2\pi)^4}(2k-q-q')^4(qq')\delta(\epsilon - cq + cq' - \epsilon_+(k-q-q'))\times$$

$$\times\left[F(k-q-q')f_+(q)f_-(q') + F(k-q-q') + f_+(q) + f_-(q')\right]. \tag{4.175}$$

Similar calculations performed for $\mathrm{Im}\Sigma^R$, Π^K and $\mathrm{Im}\Pi^R$ result in

$$2\mathrm{Im}\Sigma^R(k,\epsilon) = \frac{\pi^3}{8}\left(\frac{\hbar c\Gamma_0'}{m^*}\right)^2 \int \frac{dqdq'}{(2\pi)^4}(2k-q-q')^4(qq')\delta(\epsilon - cq + cq' - \epsilon_+(k-q-q'))\times$$

$$\times\left[f_+(q)f_-(q') + F(k-q-q')f_+(q) + F(k-q-q')f_-(q') + 1\right], \tag{4.176}$$

$$\Pi^K(q,\omega) = -i\frac{q^2}{8}\left(\frac{\pi\hbar c\Gamma_0'}{m^*}\right)^2 \int \frac{dkdq'}{(2\pi)^4}q'(2k-q+q')^2\delta(\epsilon_+(k) - \omega - cq' - \epsilon_+(k-q+q'))\times$$

$$\times\left[f_-(q')F(k)F(k-q+q') + f_-(q') + F(k) + F(k-q+q')\right], \tag{4.177}$$

$$2\mathrm{Im}\Pi^R(q,\omega) = -\frac{q^2}{8}\left(\frac{\pi\hbar c\Gamma_0'}{m^*}\right)^2 \int \frac{dkdq'}{(2\pi)^4}q'(2k-q+q')^2\delta(\epsilon_+(k) - \omega - cq' - \epsilon_+(k-q+q'))\times$$

$$\times\left[f_-(q')F(k) + F(k)F(k-q+q') - f_-(q')F(k-q+q') - 1\right]. \tag{4.178}$$

4.11 Field theory: kinetic equations II

Now that the self-energy of depletons and the polarization operator of phonons are known, we are able to write down the kinetic equations Eqs. (4.115),(4.118) for each type of excitations, with collision integrals evaluated to the lowest order in perturbation theory,

$$I_{coll}[F] = i\Sigma^K + 2F\mathrm{Im}\Sigma^R, \qquad I_{coll}[f] = i\Pi^K + 2f\mathrm{Im}\Pi^R. \tag{4.179}$$

From the calculations of Section 4.10 we see, that $\Sigma^{K/R}$ and $\Pi^{K/R}$ themselves depend on F and f in a non-trivial manner,

$$\begin{aligned} I_{coll}[F(k)] &= \frac{\pi^3}{8}\left(\frac{\hbar c\Gamma_0'}{m^*}\right)^2 \int \frac{dqdq'}{(2\pi)^4}\,(2k-q-q')^4(qq')\delta(\epsilon - cq + cq' - \epsilon_+(k-q-q'))\times \\ &\times [F(k)\,(f_+(q)f_-(q') + F(k-q-q')f_+(q) + F(k-q-q')f_-(q') + 1) - \\ &-F(k-q-q')f_+(q)f_-(q') - F(k-q-q') - f_+(q) - f_-(q')]\,, \end{aligned} \tag{4.180}$$

$$\begin{aligned} I_{coll}[f_+(q)] &= \frac{q^2}{8}\left(\frac{\pi\hbar c\Gamma_0'}{m^*}\right)^2 \int \frac{dkdq'}{(2\pi)^4}\,q'(2k-q+q')^2\delta(\epsilon_+(k) - \omega - cq' - \epsilon_+(k-q+q'))\times \\ &\times [f_-(q')F(k)F(k-q+q') + f_-(q') + F(k) + F(k-q+q') - \\ &-f_+(q)\,(f_-(q')F(k) + F(k)F(k-q+q') - f_-(q')F(k-q+q') - 1)]\,, \end{aligned} \tag{4.181}$$

so that kinetic equations

$$\frac{\partial F(k,t)}{\partial t} = I_{coll}[F(k)], \tag{4.182}$$

$$\frac{\partial f_+(q,t)}{\partial t} = I_{coll}[f_+(q)], \tag{4.183}$$

are, in fact, coupled nonlinear integro-diferrential equations. In Eqs. (4.182) and (4.183) the spatial derivatives were left out (compare to Eqs. (4.115), (4.118)) because the system is assumed spatially homogeneous, and momentum p is replaced by wavenumber, $k = p/\hbar$.

Investigating the kinetic equations with full collision integrals Eqs. (4.180), (4.181) would be a formidable numerical project. Instead we study the system close to equilibrium and assume that distribution functions F, f_+ and f_- are close to their equilibrium values

$$F_0(k) = \tanh\left(\frac{\hbar\epsilon_+(k)}{2k_BT}\right), \tag{4.184}$$

$$f_{0,+}(q) = \coth\left(\frac{c\hbar q}{2k_B T}\right), \tag{4.185}$$

$$f_{0,-}(q') = \coth\left(-\frac{c\hbar q'}{2k_B T}\right) \tag{4.186}$$

at finite temperature, and

$$F_0(k) = \mathrm{sgn}\left(\epsilon_+(k)\right), \tag{4.187}$$

$$f_{0,+}(q) = \mathrm{sgn}\left(q\right), \tag{4.188}$$

$$f_{0,-}(q') = \mathrm{sgn}\left(-q'\right), \tag{4.189}$$

at $T = 0$. We linearize equations Eqs. (4.182), (4.183) around those equilibrium distributions and write new, linear equations in terms of

$$\delta F(k) = F(k) - F_0(k), \tag{4.190}$$

$$\delta f_+(q) = f_+(q) - f_{0.+}(q), \tag{4.191}$$

$$\delta f_-(q') = f_-(q') - f_{0.-}(q'). \tag{4.192}$$

As a consistency check, one can verify that collision integrals Eqs. (4.180), (4.181), evaluated at the equilibrium distributions vanish as expected: in thermal equilibrium the distribution functions cannot change with time, and therefore the corresponding collision integrals must also vanish.

The resulting linearized equations are still coupled and still integro-differential[3]:

$$\begin{aligned}\frac{\partial \delta F(k)}{\partial t} &= -\frac{\Gamma'^2 T\hbar^2}{128\pi m^{*2}}\int dq\, q(2k+q)^4\left[\delta F(k)\left(\coth\left(\frac{\hbar cq}{2k_BT}\right)-\tanh\left(\frac{\hbar c(q+k)}{2k_BT}\right)\right)+\right.\\ &+\delta f_+(q)\left(\tanh\left(\frac{\hbar c(q+k)}{2k_BT}\right)-\tanh\left(\frac{\hbar ck}{2k_BT}\right)\right)+\\ &\left.+\delta F(k+q)\left(\coth\left(\frac{\hbar cq}{2k_BT}\right)-\tanh\left(\frac{\hbar ck}{2k_BT}\right)\right)\right], \qquad (4.193)\end{aligned}$$

$$\begin{aligned}\frac{\partial \delta f_+(q)}{\partial t} &= \frac{\Gamma'^2 T\hbar^2 q}{128\pi m^{*2}}\int dk\ (2k+q)^4\left[\delta f_+(q)\left(\tanh\left(\frac{\hbar ck}{2k_BT}\right)-\tanh\left(\frac{\hbar c(q+k)}{2k_BT}\right)\right)+\right.\\ &+\delta F(k)\left(\coth\left(\frac{\hbar cq}{2k_BT}\right)-\tanh\left(\frac{\hbar c(q+k)}{2k_BT}\right)\right)+\\ &\left.+\delta F(k+q)\left(\coth\left(\frac{\hbar cq}{2k_BT}\right)+\tanh\left(\frac{\hbar ck}{2k_BT}\right)\right)\right], \qquad (4.194)\end{aligned}$$

where the depletonic dispersion also was linearized,

$$\epsilon_+(k) \approx ck. \qquad (4.195)$$

It is therefore convenient to go a step further to the so-called collision-time approximation [92], which sets

$$\frac{\partial \delta F(k,t)}{\partial t} = -\frac{\delta F(k,t)}{\tau_F} \qquad (4.196)$$

and

$$\frac{\partial \delta f_\pm(q,t)}{\partial t} = -\frac{\delta f_\pm(q,t)}{\tau_B}, \qquad (4.197)$$

i.e. it replaces the linearized expression that mixes the bosonic and fermionic distribution functions by a single term that describes the exponential decay of the distribution function.

[3]Eqs. (4.193) and (4.194) are given without derivation here, which is similar to the derivation of τ_F^{-1} that follows.

The characteristic decay times τ_F and τ_B in Eqs. 4.196 and 4.197 can be taken from the interpretation of imaginary part of self-energy as the lifetime of quasi-particles that decay due to collisions, namely

$$\tau_F^{-1}(k) = -2\mathrm{Im}\Sigma^{\mathrm{R}}(k, \epsilon_+(k)), \tag{4.198}$$

$$\tau_B^{-1}(q) = -2\mathrm{Im}\Pi^{\mathrm{R}}(q, cq). \tag{4.199}$$

According to Eq. (4.176) the retarded component of the on-shell fermionic self-energy at wavenumber k is given by the following expression

$$\begin{aligned}\mathrm{Im}\Sigma^{\mathrm{R}}(k, E(k)) &= \frac{\hbar^2 c^2 \left(\Gamma_0'\right)^2}{128\pi m^{*2}} \int dq dq'\, qq' \left(2k - q - q'\right)^4 \delta\left(\epsilon_+(k) - cq + cq' - \epsilon_+\left(k - q - q'\right)\right) \times \\ &\times \left[f_{0,+}\left(cq\right) f_{0,-}\left(-cq'\right) + F_0\left(\epsilon_+(k) - cq + cq'\right) f_{0,+}\left(cq\right) + \right. \\ &\left. + F_0\left(\epsilon_+(k) - cq + cq'\right) f_{0,-}(-cq')\right]. \end{aligned} \tag{4.200}$$

To integrate out q' in Eq. (4.200) we realize that

$$\delta\left(\epsilon_+(k) - cq + cq' - \epsilon_+(k - q - q')\right) \tag{4.201}$$

restricts wavenumber of the left-moving phonon to a very small value proportional to the curvature of the fermionic spectrum:

$$\delta\left(\epsilon_+(k) - cq + cq' - \epsilon_+(k - q - q')\right) \approx \frac{1}{2c}\delta\left(q' - \frac{\hbar q(q - 2k)}{4cm^*}\right) \approx \frac{1}{2c}\delta\left(q'\right), \tag{4.202}$$

so that it can be approximated by delta function at $q' = 0$. The next observation to make is that since the integral contains a combination

$$q'\delta(q') \tag{4.203}$$

then all the terms in square brackets that are regular in q' will vanish, so only the terms containing $f_{0,-}$, which is divergent at small q', need to be kept:

$$f_{0,-}(-cq') \approx -\frac{2k_BT}{\hbar cq'}. \tag{4.204}$$

Employing Eq. (4.204) we obtain:

$$\mathrm{Im}\Sigma^{\mathrm{R}}(k,E(k)) \approx -\frac{\hbar c^2\Gamma^2 k_BT}{128c^2\pi m*^2}\int dq\, q\,(2k-q)^4\left[f_{0,+}\left(cq\right)+F_0\left(ck-cq\right)\right],$$

$$\tau_F^{-1}(k) = -2\mathrm{Im}\Sigma^{\mathrm{R}}(k,E(k)) \approx \frac{\hbar c^2\left(\Gamma_0'\right)^2 k_BT}{64c^2\pi m*^2}\int dq\, q\,(2k-q)^4\left[\coth\left(\frac{\hbar cq}{2k_BT}\right)-\tanh\left(\frac{\hbar cq-\hbar ck}{2k_BT}\right)\right]. \tag{4.205}$$

If we are interested in behavior of slow depletons, we can evaluate the limit of τ_F^{-1} at low wave numbers by setting $k=0$ in Eq. (4.205):

$$\begin{aligned}\tau_F^{-1} = \tau_F^{-1}(0) &= \frac{\hbar c^2\left(\Gamma_0'\right)^2 k_BT}{64c^2\pi m*^2}\int dq\, q^5\left[\coth\left(\frac{\hbar cq}{2k_BT}\right)-\tanh\left(\frac{\hbar cq}{2k_BT}\right)\right]\\ &= \frac{\hbar c^2\Gamma^2 k_BT}{64c^2\pi m*^2}\left(\frac{2k_BT}{\hbar c}\right)^6\int_{-\infty}^{\infty} du\, u^5\left[\coth(u)-\tanh(u)\right].\end{aligned} \tag{4.206}$$

Evaluating the integral on the right hand side of Eq. (4.206),

$$\int_{-\infty}^{\infty} du\, u^5\left[\coth(u)-\tanh(u)\right] = \frac{\pi^6}{64}, \tag{4.207}$$

the depleton lifetime is obtained:

$$\tau_F^{-1} = \frac{\left(\Gamma_0'\right)^2\pi^5\left(k_BT\right)^7}{128c^6\hbar^5 m*^2}. \tag{4.208}$$

At low momenta the depleton decay time is k-independent, a clearly non-hydrodynamic result. τ_F^{-1} is a new feature that appears due to inclusion of sharp field configurations into the model, it is a new time scale that is crucial to describing the dynamical correlators such as $S(q,\omega)$, as will be discussed in the next Section.

For the sake of completeness we also compute τ_F^{-1} at $T=0$, although we we are interested in finite-temperature dynamics in this Thesis. The difference between $T=0$ and $T>0$ cases is that equilibrium distributions are given by signum functions, Eqs. (4.187)-(4.189), and the computations need to be modified accordingly. Eq. (4.200) becomes

$$\begin{aligned}\tau_F^{-1}(k) &= \frac{\Gamma'^2 c^2\hbar^2}{128\pi m^{*2}}\int dqdq'\, qq'(2k-q-q')^4\frac{1}{2c}\delta\left(q'-\frac{q(q-2k)}{4m^*c}\right)\times \\ &\times\left[\mathrm{sgn}(q)\mathrm{sgn}(-q')+\mathrm{sgn}(k-q)\mathrm{sgn}(q)+\mathrm{sgn}(k-q)\mathrm{sgn}(-q')+1\right]. \quad (4.209)\end{aligned}$$

The expression in square brackets can be rearranged

$$\mathrm{sgn}(q)\mathrm{sgn}(-q')+\mathrm{sgn}(k-q)\mathrm{sgn}(q)+\mathrm{sgn}(k-q)\mathrm{sgn}(-q')+1=$$

$$=\mathrm{sgn}(-q')\left[\mathrm{sgn}(q)-\mathrm{sgn}(q-k)\right]+\left[1-\mathrm{sgn}(q-k)\mathrm{sgn}(q)\right]. \quad (4.210)$$

Both expressions in square brackets are non–zero only on the segment $0<q<k$, and evaluate to 2 there.

$$\mathrm{sgn}(-q')=\mathrm{sgn}\left((2k-q)q\right)=1 \quad (4.211)$$

in the same region. This restricts the domain of integration and the decay rate evaluates to

$$\tau_F^{-1}(k)=\frac{\Gamma'^2c^2\hbar^2}{128\pi m^{*2}}\int_0^k dq\frac{q^2(2k-q)^5}{2m^*c^2}=\frac{73\hbar^2\Gamma'^2k^8}{14336\pi m^{*3}}, \quad (4.212)$$

where q' was neglected in $(2k-q-q')$. Scaling with k^8 in Eq. (4.212) is in agreement with

results obtained by Khodas, Pustilnik, Kamenev and Glazman [94], and later by Matveev and Furusaki [95].

Similar calculations can be performed to find $\tau_B^{-1}(q)$ both at $T > 0$ and $T = 0$, however from Eq. (4.163) one can tell that the polarization operator is proportional to q^3 [79], and therefore phonons in the modified theory containing depletons behave hydrodynamically, long wavelength excitations have very large lifetime. That is, in the complete absence of depletons we would expect the hydrodynamic description to hold.

4.12 $S(q, \omega)$ of the 1D system

In this section we follow up on findings of Sec. 4.11 and discuss the (semiclassical) behavior of $S(q, \omega)$ in different frequency regimes at finite temperature.

We have previously found that sharp density and velocity field configurations, that we call either impurities or depletons, show non-hydrodynamic behavior in the sense that for low momenta/wave numbers the lifetime of sharp excitations is finite,

$$\tau_F^{-1}(k) \to const, \qquad k \to 0, \tag{4.213}$$

see Eq. (4.208). On the contrary, for phonons the usual hydrodynamic behavior holds [96],

$$\tau_B^{-1}(k) \to 0, \qquad k \to 0. \tag{4.214}$$

From this we must conclude that depletons introduce a new timescale into the problem,

$$\tau_F = \frac{128 c^6 \hbar^5 m^{*2}}{\left(\Gamma_0'\right)^2 \pi^5 \left(k_B T\right)^7}. \tag{4.215}$$

The dynamics of the system is qualitatively different for $t < \tau_F(0)$, when the depletons are still present on top of phonons after the initial perturbation, with phonons providing the thermal bath, and for $t > \tau_F(0)$ when all the depletons have decayed and the physics of the system is that of weakly interacting phonons, Section (4.3). Inverting from the (x,t) space to (q,ω) space we may say that for $\omega < \tau_F^{-1}(0)$ the impurities no longer matter and hydrodynamics is applicable. This is not that surprising, after all the hydrodynamics by definition is description of the system in terms of low-energy, long-wavelength excitations [81]. What is new here is the exact bound on the smallness of the frequency for hydrodynamics to be applicable. At low frequencies,

$$\omega \ll \frac{\left(\Gamma_0'\right)^2 \pi^5 \left(k_B T\right)^7}{128 c^6 \hbar^5 m^{*2}}, \tag{4.216}$$

the depletons are not important and Andreev's self-consistent approach to non-linear hydrodynamics, mentioned in Section 4.3, may be used to calculate $S(q,\omega)$. Hence, at $T > 0$ and $\omega\tau_F(0) \ll 1$ the Dynamical Structure Factor $S(q,\omega)$ is represented by two peaks centered at $\omega = \pm cq$ that have some universal shape with width scaling as

$$\delta\omega \sim q^{3/2}, \tag{4.217}$$

the Andreev's result [43, 83] that has also recently been discussed in the context of numerical simulations of 1D Gross-Pitaevskii equation [44].

On the other hand, to describe the system at frequencies larger than τ_F^{-1} we need to abandon hydrodynamics altogether and treat the system as a collection of fermions (depletons), interacting indirectly through collisions with phononic bath (see Fig. 4.7). To find $S(q,\omega)$ in this regime we use the kinetic equations Eq. (4.196) and follow the semiclassical treatment of [92]. Namely, we introduce the generalized susceptibility $\alpha(q,\omega)$, the function that connects

the Fourier components of external potential $U(q,\omega)$ to the density fluctuation $\delta\rho(q,\omega)$,

$$\delta\rho(q,\omega) = \alpha(q,\omega)U(q,\omega), \tag{4.218}$$

and use the fluctuation-dissipation theorem [63, 97] to connect $S(q,\omega) = \langle\delta\rho\delta\rho\rangle(q,\omega)$ and $\alpha(q,\omega)$. In the classical limit this connection is:

$$S(q,\omega) = \frac{2k_BT}{\omega}\mathrm{Im}\alpha(q,\omega). \tag{4.219}$$

We also introduce generalized susceptibility $\chi(q,\omega;p)$ that connects U and Fourier components of

$$\delta F(x,p,t) \to \delta F(q,\omega;p), \tag{4.220}$$

such that

$$\delta F(q,\omega;p) = \chi(q,\omega;p)U(q,\omega) \tag{4.221}$$

and

$$\alpha(q,\omega) = \int \chi(q,\omega;p)\,dp. \tag{4.222}$$

By Fourier transforming the linearized kinetic equation in the collision time approximation,

$$\frac{\partial\delta F}{\partial t} + v(p)\frac{\partial\delta F}{\partial x} - \frac{\partial U}{\partial r}\frac{\partial F_0}{\partial p} = -\frac{\delta F}{\tau_F}$$

$$\downarrow$$

$$-i\omega\delta F(q,\omega;p) + \left(c + \frac{p}{m^*}\right)iq\delta F(q,\omega;p) - iqU(q,\omega)\frac{\partial F_0}{\partial p} = -\frac{\delta F(q,\omega;p)}{\tau_F} \tag{4.223}$$

and dividing it by $U(q,\omega)$, we obtain the equation for $\chi(q,\omega;p)$:

$$\chi(q,\omega;p) = iq\frac{\partial F_0}{\partial p}\left[i\left(\left(c+\frac{p}{m^*}\right)q-\omega\right)+\frac{1}{\tau_F}\right]^{-1}. \tag{4.224}$$

By now connecting Eqs. (4.219), (4.222) and (4.224), we obtain semiclassical expression for $S(q,\omega)$,

$$S(q,\omega) = \frac{2k_BT}{\omega}\int\frac{dp}{2\pi}\,\mathrm{Im}\left(iq\frac{\partial F_0}{\partial p}\left[i\left(\left(c+\frac{p}{m^*}\right)q-\omega\right)+\frac{1}{\tau_F}\right]^{-1}\right),$$

$$S(q,\omega) = \frac{2k_BTq}{\omega}\int\frac{dp}{2\pi}\,\frac{\tau_F\partial F_0/\partial p}{1+\tau_F^2\left(\frac{pq}{m^*}-(\omega-cq)\right)^2}. \tag{4.225}$$

Eq. (4.225) is peaked at $\omega = cq$, therefore we are justified to set $q/\omega = 1/c$ in the prefactor. Moreover, since most contribution to the integral comes from small p, we can approximately consider the dispersion to be linear, $E_+(p) \approx cp$, so that

$$F_0 \approx \tanh\left(\frac{cp}{2k_BT}\right) \tag{4.226}$$

and

$$\frac{\partial F_0}{\partial p} \approx \frac{c}{2k_BT\cosh^2\left(\frac{cp}{2k_BT}\right)}, \tag{4.227}$$

so that the expression for DSF becomes

$$S(q,\omega) = \int\frac{dp}{2\pi}\,\frac{\tau_F\cosh^{-2}\left(cp/2k_BT\right)}{1+\tau_F^2\left(\frac{pq}{m^*}-(\omega-cq)\right)^2}. \tag{4.228}$$

The expression Eq. (4.228) is correct up to two numerical prefactors. One of those is 1/2 and it comes from the fact that F is not the actual distribution function, but is related to it via Eq. (4.111), so that

$$\delta F(x,p;t) = -2\delta n(x,p;t). \tag{4.229}$$

Another prefactor is the Luttinger parameter K and its appearance is somewhat more subtle. In Eqs. (4.10) and (4.11) we have made transition to the chiral variables χ_+ and χ_-. Apart from making the Lagrangian separable, χ_+ and χ_- have as well eliminated the Luttinger parameter K from further consideration. However, the actual density is still related to the density of fields χ and density of depletons formed on top of those via multiplication by $\sqrt{K}$,

$$\delta\rho \sim \sqrt{K}\delta\rho_\chi, \tag{4.230}$$

compare this also to Eq. (4.5), where the factor of K appears in front of the correlator of χ-s. Therefore the correct DSF in the "depletonic" regime of a 1D system is given by

$$S(q,\omega) = \int \frac{dp}{4\pi} \frac{K\tau_F \cosh^{-2}\left(cp/2k_BT\right)}{1+\tau_F^2\left(\frac{pq}{m^*}-(\omega-cq)\right)^2}. \tag{4.231}$$

Taking into account also the left-moving impurities, the result similar to Eq. (4.231) can be obtained, but centered on the $\omega = -cq$ line.

Finally, we notice that Eq. (4.231) under the integral contains a product of Lorentzian with width $m^*/\tau_F q$ and centered at $p = m^*/q(\omega - cq)$,

$$\frac{1}{1+\tau_F^2\left(\frac{pq}{m^*}-(\omega-cq)\right)^2},$$

and inverse $\cosh^2$ with width Tk_B/c and centered at $p = 0$, Eq. (4.227). For $q > q_c = m^*c/k_BT\tau_F$, the Lorentzian is the narrowest of the two and can be treated as a delta function. This results in

$$S(q,\omega) = \frac{Km^*}{4q}\cosh^{-2}\left(\frac{m^*c}{2k_BTq}(\omega-cq)\right), \quad q > q_c. \tag{4.232}$$

In this regime the width of the DSF is the just the "fermionic" result of Section 4.4,

$$\delta\omega = \frac{2k_B T q}{m^* c}, \tag{4.233}$$

but featuring the effective mass of the excitation m^*, instead of the bare mass of a particle in the liquid. This reflects the fact that Eq. (4.233) is a general result, valid for all values of K. In the opposite limit, $q < q_c$, the Lorenzian is a smooth function compared to the quickly decaying inverse $\cosh^2$, and can be evaluated at $p = 0$ and taken out of the integral, resulting in

$$S(q,\omega) = \frac{K k_B T}{\pi c} \frac{\tau_F}{1 + \tau_F^2 \left(\omega - cq\right)^2}, \quad q < q_c.$$

In this regime the width of DSF is given by

$$\delta\omega = \tau_F^{-1},$$

the result determined by collisions with the bath of phonons.

4.13 Experimental relevance and discussion

The findings of Section 4.12, and indeed of Chapter 4 are best summarized by the Figure 4.10. Including sharp configurations of density/velocity fields into consideration has introduced the timescale τ_F, Eq. (4.215) which defines the region of validity of hydrodynamics. In particular, τ_F is inversely proportional to the backscattering amplitude squared,

$$\tau_F \propto \frac{1}{\left(\Gamma_0'\right)^2}.$$

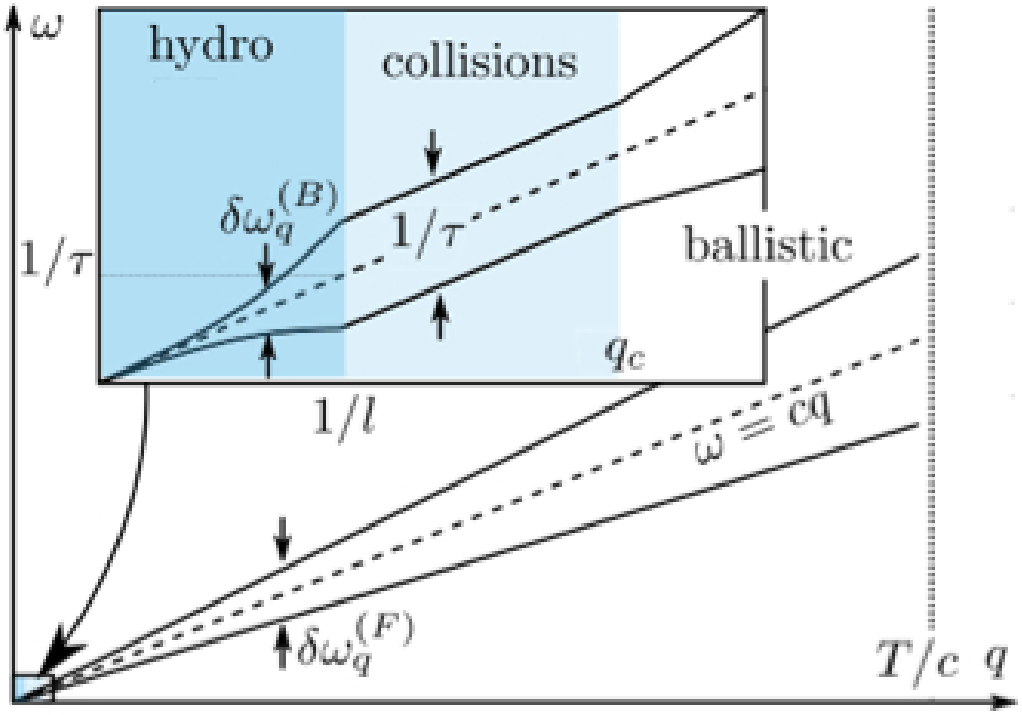


Figure 4.10: Figure taken from [79] that represents different regimes of the dynamical structure factor $S(q,\omega)$ on the (q,ω) plane. The ballistic regime corresponding to dynamics of depletons, Eq. (4.232), dominates the plot, while hydrodynamics holds for frequencies smaller that τ_F^{-1}.

This means that for integrable models, where no backscattering of depletons is taking place and $\Gamma_0' = 0$,

$$\tau_F^{-1} = 0,$$

and hydrodynamics is **not** valid for **any** values of q, ω. Notice how this resolves the controversy raised at the end of Section 4.4. In the Tonks-Girardeau limit, $K \to 1$, the system approaches integrable point - the free fermions. Therefore the hydrodynamics is not expected to be valid and the approach of [43, 83] gives incorrect results.

To conclude this discussion we demonstrate that depletons are something potentially observable in experiments. In order to do so we estimate the typical value of τ_F^{-1} using the values

of experimental parameters currently in use. We use the expression found in [98] for α of Eq. (4.92),

$$\alpha = 12 \ln\left(\frac{4}{3}\right) \hbar a^2 \omega_\perp,$$

where a is the scattering length and $\omega_\perp$ is the frequency of transverse confinement that restricts the system to 1D. Using also the formula for effective mass Eq. (4.79), the equation Eq. (4.215) is recast in the form

$$\frac{\hbar}{\tau_F mc^2} = \frac{A}{K^3} \left(\frac{mc^2}{\hbar\omega_\perp}\right)^2 \left(\frac{k_B T}{mc^2}\right)^7. \tag{4.234}$$

$A = 9\pi^7 \left(\ln(4/3)\right)^2 / 2^{15} \approx 0.07$, and for experiment described in [8] $K \sim 10$, $mc^2/\hbar \sim 5.5kHz$ and $\omega_\perp \sim 66kHz$ which results in τ_F of the order 10 seconds even for $k_B T/mc^2 = 1$. Because Eq. (4.234) depends on temperature as T^7 decreasing the temperature can lead to even longer lifetimes, of order of days, making the non-hydrodynamic result the only one likely to be observed.

Chapter 5

Conclusions

We have started this Thesis with investigating the momentum-momentum correlator $\mathcal{G}(k,k') = \langle \delta n_k \delta n_{k'} \rangle$ in different regimes of a weakly interacting Bose gas. For the limiting case of zero interactions this correlator is proportional to a delta function, $\mathcal{G}(k,k') \propto \delta_{k,k'}$, a result which is expected. Indeed, in the absence of collisions momenta of every particle are separately conserved, therefore there can be no correlations between different momentum values. Finite interaction would leave its imprint in the form of non-zero non-diagonal part of $\langle \delta n_k \delta n_{k'} \rangle$ and this motivated us to investigate the behavior of the correlator in a 1D interacting system. For 1D dimensionality matters, too: it is known that at low temperatures the 3D interacting Bose gas forms a phase-coherent condensate (Bose-Einstein condensate), plus the cloud of excitations on top of it [19], however in 1D the phase transition associated with the onset of such condensation is not possible [16, 17], and the correlation functions must reflect this fact.

Allowing for repulsive interatomic interactions **and** assuming presence of the condensate, the Bogoliubov theory [66, 67] results in additional line of opposite momenta correlations, $\mathcal{G}(k,k') = ...\delta_{k,k'} + ...\delta_{k,-k'}$. For example, at $T = 0$ the degree of correlations between equal and opposite momenta is the same, due to the nature of the Bogoliubov ground state [27].

In 1D the Bogoliubov theory is still valid on the length-scales smaller than the coherence length l_φ of the gas, but experimental restrictions mean that sizes of the atomic gas samples available are usually much larger than l_φ in those samples, which implies that Bogoliubov's condensation approximation cannot be valid and corrections must arise. We consider such regime of the phase-fluctuating quasicondensate in Section 2.3 of Chapter 2. The Luttinger Liquid description is perfectly suitable for this task, as it easily allows to implement the conditions of quasicondensate regime: phase fluctuations are allowed but no fluctuations of density take place. The result obtained for the quasicondensate regime is indeed very different from the prediction of Bogoliubov theory, resulting in non-zero $\mathcal{G}(k,k')$ on the entire (k,k') plane, and even in regions of anticorrelation, $\mathcal{G}(k,k') < 0$.

To investigate the crossover from regimes of the phase-fluctuating quasicondensate to the strongly degenerate Bose gas as interaction strength g, chemical potential μ and temperature T are varied we apply the classical field approximation, see Section 2.4, that maps the problem of computing field correlators in 1D field theory to the problem of solving the Schrodinger equation in 2D quantum mechanics. Our numerical procedure allows to approximately obtain the momentum-momentum correlator for a broad range of parameter

$$\eta = \left(\frac{\hbar^2}{m g^2 k_B^2 T^2}\right)^{1/3} \mu, \tag{5.1}$$

also giving the quasicondensate and strongly degenerate results in the limits $\eta \to \infty$ and $\eta \to -\infty$, respectively.

The numerical procedure involved in finding the $\mathcal{G}(k,k')$ for the arbitrary value of parameter η, Eq. (5.1), turns out to be relevant to the problem of particle number distribution in a pixel of a weakly interacting sample of 1D gas, which we discuss in Chapter 3. The (approximate) particle number distribution allows to compute all moments of the particle number on a

segment.

In Chapter 4 of this Thesis we return to the discussion that was started in the Introduction about applicability of hydrodynamics to 1D systems. Great success of the Tomonaga-Luttinger model in describing various static correlation functions (see e.g.. [39]), and in particular its successful description of the momentum-momentum correlator of the quasicondensate in Chapter (2) has to be weighted against its failure to correctly describe the Dynamical Structure Factor in the Tonks-Girardeau gas, see Section 4.4. It turns out that dynamical correlation functions cannot be adequately described by hydrodynamic theories in the entire (q, ω) plane, since hydrodynamics intentionally ignores sharp configurations of density / velocity fields, and therefore misses the exact 1D excitation spectrum, consisting of particle-hole excitations (compare to Lieb-Liniger model). We find a way of taking such sharp configurations (which we also call impurities, or depletons) into account and find that a new time-scale τ_F emerges, that corresponds to the lifetime of such sharp excitations. τ_F is finite for depletons of arbitrary momentum which is a non-hydrodynamic result. Such effect cannot be obtained by "collectivizing" the degrees of freedom into phonons that always have $\tau_{phon} \to \infty$ as their momentum goes to zero (i.e. long wavelength phonons decay infinitely slowly).

Presence of the finite lifetime τ_F means that hydrodynamics must not be used to compute the dynamical correlators at smaller times. Instead one should use the picture of sharp excitations / impurities as fermions in the phononic bath, which results in an entirely different behavior of the correlators. On the other hand, when all the impurities formed during the initial perturbation of the system have decayed at $t > \tau_F$ so that only phonons remain the hydrodynamics is applicable. In particular, for DSF this means that self-consistent result of [43, 83] holds.

The threshold value of time / inverse frequency τ_F contains the parameter Γ_0', which corresponds to the rate of backscattering of impurities off phonons. For integrable systems the impurities are exact particle / hole like excitations that do not backscatter, but rather follow the

Schrodinger evolution, $\Gamma_0' = 0$. Therefore for integrable systems the lifetime τ_F is infinite and hydrodynamics is never applicable. This explains inapplicability of the Luttinger Liquid model, even with higher order corrections to the dispersion, to the case $K = 1$, which corresponds to free fermions, an integrable model.

It is also interesting to notice that τ_F contains strong temperature dependence, T^7, which makes it a promising endeavor to design experiments to observe the mentioned non-hydrodynamic behavior in 1D ultracold atoms. Indeed, by estimating the τ_F based on the values of experimental parameters currently used we obtain time of the order of days. This makes hydrodynamic region of validity $\omega < \tau_F^{-1}$ tiny and opens up the possibility to experimentally observe 1D dynamic beyond hydrodynamics.

Appendix A

DSF of free spinless fermions

Consider free spinless fermions with quadratic dispersion with Hamiltonian

$$H = \sum_k \frac{\hbar^2 k^2}{2m} c_k^\dagger c_k \tag{A.1}$$

and kept at temperature T and chemical potential μ. First we remind ourselves that

$$\begin{aligned} S(q,\omega) &= \int dx dt\, e^{-i(qx-\omega t)} \left(\langle \rho(x,t)\rho(0,0)\rangle - \rho_0^2 \right) \\ &= \int dt\, e^{i\omega t} \left(\left\langle e^{iHt/\hbar} \left(\int dx\, \rho(x) e^{-iqx} \right) e^{-iHt/\hbar} \rho(0) \right\rangle - 2\pi\delta(q)\rho_0^2 \right), \end{aligned} \tag{A.2}$$

where

$$\rho(x) = c^\dagger(x)c(x) = \frac{1}{L}\sum_{k,k'} c_k^\dagger c_{k'} e^{-i(k-k')x}, \tag{A.3}$$

and

$$\rho_q = \int dx\, \rho(x) e^{-iqx} = \sum_k c_k^\dagger c_{k+q}. \tag{A.4}$$

This allows to rewrite Eq. (A.2) as

$$S(q,\omega) = \int dt\, e^{i\omega t} \left(\frac{1}{L} \sum_{k,k',k''} \left\langle e^{iHt/\hbar} c_k^\dagger c_{k+q} e^{-iHt/\hbar} c_{k'}^\dagger c_{k''} \right\rangle - 2\pi\delta(q)\rho_0^2 \right). \tag{A.5}$$

We now observe that the following commutation relation is valid for free fermions:

$$c_k e^{-iHt/\hbar} = e^{iE_k t/\hbar} e^{-iHt/\hbar} c_k, \tag{A.6}$$

where $E_k = \hbar^2 k^2/2m$ is the energy of the particle with momentum k. Eq. (A.6) can be justified by applying it to any eigenstate of the Hamiltonian Eq. (4.44). In a similar manner one can derive the relation for the creation operator:

$$c_k^\dagger e^{-iHt/\hbar} = e^{-iE_k t/\hbar} e^{-iHt/\hbar} c_k^\dagger. \tag{A.7}$$

Eqs. (A.6) and (A.7) allow us to commute $e^{-iHt/\hbar}$ in Eq. (A.5) past the operators on its left to obtain

$$S(q,\omega) = \int dt\, e^{i\omega t} \left(\frac{1}{L} \sum_{k,k',k''} e^{i(E_{k+q}-E_k)t/\hbar} \left\langle c_k^\dagger c_{k+q} c_{k'}^\dagger c_{k''} \right\rangle - 2\pi\delta(q)\rho_0^2 \right). \tag{A.8}$$

Finally, the average of the four operators in Eq. (A.8) can be computed with the aid of Wick's theorem

$$\left\langle c_1 c_2 c_3^\dagger c_4^\dagger \right\rangle = -\left\langle c_1 c_3^\dagger \right\rangle \left\langle c_2 c_4^\dagger \right\rangle + \left\langle c_1 c_4^\dagger \right\rangle \left\langle c_2 c_3^\dagger \right\rangle, \tag{A.9}$$

the anticommutation relation

$$\left\{ c_1, c_2^\dagger \right\} = \delta_{1,2}, \tag{A.10}$$

and the averages

$$\left\langle c_1^\dagger c_2 \right\rangle = n_F(E_1)\delta_{1,2}, \qquad \left\langle c_2 c_1^\dagger \right\rangle = \left(1 - n_F(E_1)\right)\delta_{1,2}, \tag{A.11}$$

where $n_F(E)$ is the Fermi function

$$n_F(E) = \frac{1}{e^{\beta(E-\mu)}+1}. \tag{A.12}$$

Applying this machinery, we get

$$\begin{aligned}\left\langle c_k^\dagger c_{k+q} c_{k'}^\dagger c_{k''} \right\rangle &= \left\langle c_k^\dagger c_{k+q} \right\rangle \delta_{k',k''} - \left\langle c_{k''} c_{k'}^\dagger \right\rangle \delta_{k,k+q} + \\ &+ \left\langle c_{k+q} c_{k'}^\dagger \right\rangle \delta_{k,k''} + \left\langle c_{k+q} c_k^\dagger \right\rangle \left\langle c_{k''} c_{k'}^\dagger \right\rangle - \left\langle c_{k+q} c_{k'}^\dagger \right\rangle \left\langle c_{k''} c_k^\dagger \right\rangle, \end{aligned} \tag{A.13}$$

which after fermionic averaging Eq. (A.11) becomes

$$\begin{aligned}\left\langle c_k^\dagger c_{k+q} c_{k'}^\dagger c_{k''} \right\rangle &= n_F(E_k)\delta_{k,k+q}\delta_{k',k''} - \left(1 - n_F(E_{k'})\right)\delta_{k',k''}\delta_{k,k+q} + \\ &+ \left(1 - n_F(E_{k'})\right)\delta_{k',k+q}\delta_{k,k''} + \left(1 - n_F(E_k)\right)\left(1 - n_F(E_{k'})\right)\delta_{k,k+q}\delta_{k',k''} - \\ &- \left(1 - n_F(E_k)\right)\left(1 - n_F(E_{k'})\right)\delta_{k',k+q}\delta_{k,k''}. \end{aligned} \tag{A.14}$$

Contracting the similar terms, we obtain

$$\left\langle c_k^\dagger c_{k+q} c_{k'}^\dagger c_{k''} \right\rangle = n_F(E_k) n_F(E_{k'})\delta_{k',k''}\delta_{k,k+q} + \left(n_F(E_k) - n_F(E_k) n_F(E_{k'})\right)\delta_{k',k+q}\delta_{k,k''} \tag{A.15}$$

and

$$\begin{aligned}\int dt\, e^{i\omega t} 2\pi\delta(q)\rho_0^2 + S(q,\omega) &= \frac{1}{L}\int dt\, e^{i\omega t}\sum_{k,k',k''} e^{i(E_{k+q}-E_k)t/\hbar} n_F(E_k) n_F(E_{k'})\delta_{k',k''}\delta_{k,k+q} + \\ &\quad + \frac{1}{L}\int dt\, e^{i\omega t}\sum_{k,k',k''} e^{i(E_{k+q}-E_k)t/\hbar}\left(n_F(E_k) - n_F(E_k)n_F(E_{k'})\right)\delta_{k',k+q}\delta_{k,k''} \\ &= \frac{\delta_{q,0}}{L}\int dt\, e^{i\omega t}\sum_{k,k'} n_F(E_k)n_F(E_{k'}) + \\ &\quad + \frac{1}{L}\int dt\, e^{i\omega t}\sum_{k} e^{i(E_{k+q}-E_k)t/\hbar} n_F(E_k)\left(1-n(E_{k+q})\right). \end{aligned} \tag{A.16}$$

The first terms on the right and left hand sides of Eq. (A.16) cancel and the following expression is obtained for $S(q,\omega)$:

$$\begin{aligned} S(q,\omega) &= \frac{1}{L}\int dt\, e^{i\omega t}\sum_{k} e^{i(E_{k+q}-E_k)t/\hbar} n_F(E_k)\left(1-n(E_{k+q})\right) \\ &= \frac{1}{2\pi}\int dt dk\, e^{i\omega t} e^{-i(E_k-E_{k+q})t/\hbar} n_F(E_k)\left(1-n(E_{k+q})\right), \end{aligned} \tag{A.17}$$

where we have made the transition from the sum to the integral. Developing Eq. (A.17) further, we obtain

$$\begin{aligned} S(q,\omega) &= \int dk\,\delta\left(\omega - \frac{E_k - E_{k+q}}{\hbar}\right) n_F(E_k)\left(1-n(E_{k+q})\right) \\ &= \frac{m}{\hbar q}\int dk\,\delta\left(k + \frac{q}{2} + \frac{m\omega}{\hbar q}\right) n_F\left(E_k\right)\left(1-n(E_{k+q})\right), \end{aligned} \tag{A.18}$$

which after evaluation of the delta function results in Eq. (4.45).

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